Notes for the Teacher

Students explore translations as functions. They construct a translation function, label the variables using function notation, and trace and compare the paths of the variables. They restrict the domain to reveal the corresponding range, and they change the length and angle of the translation vector to create different members of the translation family. Finally, students solve translation challenges by figuring out the vector (in polar, rectangular, or two-point form) needed to match an existing rotation.

Note that the material in this activity is also covered in the activity *Family Relationships—Rotation, Dilation, and Translation Families.* This activity contains more explicit student directions (following the pattern established in the prerequisite activity on the reflection family) and covers only the translation family. The *Family Relationships* activity is more ambitious, expecting students to work independently as they explore and compare the rotation, dilation, and translation families. You should use that activity if your students are prepared for its more independent approach.

This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Construct an independent variable point *x* and translate it by a polar vector.
- Label the dependent variable using function notation.
- Drag the independent variable while tracing both variables, and describe the variables' relative motion (their *covariation*) and the relationship between their traces (their *correspondence*).
- Translate an independent point using a polar vector defined by a distance parameter and an angle parameter.
- Adjust the parameters to create different members of the translation family.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Restrict the domain of the independent variable to the border of a polygon, and observe and describe the resulting range by dragging the independent variable.
- Identify any fixed points of the function, and explicitly compare the relative motion (both speed and direction) of the independent and dependent variables.
- Solve challenges that involve finding a translation vector in polar, rectangular, or two-point form given an object and its translated image.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

- *Reflection Challenges—The Reflection Family* (Prerequisite)
- *ID the Suspects—Identify Functions* (Recommended)
- *Family Resemblances—Identify Function Families* (Recommended)
- *Rotation Challenges—The Rotation Family* (Recommended)

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together in developing students' conceptual understanding of functions.

High Cognitive Demand: This activity provides several tasks for which there is no cut-and-dried procedure for students to follow. Though the worksheet provides fairly explicit directions to help students perform the initial construction, the questions it asks require experimentation, inquiry, and analysis.

Mathematical Habits of Mind, Reasoning and Sense Making: Students construct and investigate mathematical objects, and are challenged to answer questions that require tinkering and analysis to understand the behavior of these objects.

Inquiry: The body of the activity supports student inquiry, and the worksheet and challenges contain probing questions that require students to manipulate, observe, and analyze. The probing questions include: What happens when you drag the independent variable toward the dependent variable? What's the relationship between translation and congruence?

Cooperative Learning: Students work in pairs during the exploration portion of the activity, and exchange roles between driving (using the mouse and keyboard) and coaching/recording. Expect students to work purposefully in pairs, to coach each other, and to discuss every part of the activity with their partners. The members of each pair should share the construction and manipulation steps equally, so that each student is fully involved in both creating the function and working with it.

Assessment: You should engage in formative assessment by visiting and questioning student pairs during the exploration phase. You can use the summary discussion to elicit students' understandings, confusions, and questions. The last page of the worksheet is an exit ticket.

Differentiation: The worksheet comes in a long form (with more explicit instructions) and a short form (for students with more experience with Sketchpad). It also includes an optional Answer sheet on which students can write their answers. The concluding Translation Challenges entail varying levels of difficulty, and are useful for furthering students' understanding of the Translation Family and of function concepts in general.

Questioning and Discourse: Most discourse will take place between team members during inquiry, so it's important to encourage team members to describe their observations to each other and to discuss their answers to the questions. The questions on the worksheet, along with teacher observations of the work of different teams, should guide the summary discussion. Since the worksheet includes explicit directions for completing the construction, it's important to ask questions that focus students' thinking on the big ideas, both while they are working in pairs and during the whole-class discussion.

Instructional Strategies: By varying x, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

The time students will require for this activity can vary significantly depending on their mathematical background and Sketchpad experience. Because the concluding class discussion is critical, you should be prepared to postpone the challenges contained in **Translation Challenges.gsp** to another day. While you can assign the Translation Challenges as homework exercises, it's preferable to have students work cooperatively in class while you observe their investigations.

Depending on available time and your students' proficiency with Sketchpad, it may be possible for them to do both this activity and the Dilation Challenges activity on the same day, and then do the challenges related to these two activities on a second day.

The three stages of the activity are described in more detail below, in the sections on Launch, Inquiry, and Summary.

Worksheet: The worksheet comes in two forms, both of which contain the same questions. The longer form is useful for students who are relatively new to Sketchpad, as it provides explicit steps to follow. If students have experience with Sketchpad, the shorter form is preferable, as it expects more initiative from students and it concentrates more on the mathematics to be explored and less on technicalities. (Giving students unnecessarily explicit directions risks miring students in details, discouraging their thoughtfulness and creativity.) Some teachers may choose to provide one short form and one long form to each pair of students, telling them to work from the short form, referring to the details in the long form as needed.

Answer Sheet: Many teachers prefer their students to write their answers on their own paper. Alternatively, you might ask students to submit their answers electronically, or you can use the provided answer sheet. If you assign electronic submission, tell students that they can include screen captures by choosing **Edit | Select All** and then **Edit | Copy** in Sketchpad. They can paste the resulting image into a word processor, email, or other document. (The image includes traces and is cropped to the window border, so that students can easily control the area being copied and pasted.)

Transfer of Learning: To get the most value from this activity, students must connect what they've learned to other representations of functions. By dragging the independent variable x in this activity, they can relate its smooth motion to the continuous variation of a numeric variable. Moreover, comparing the relative speed and direction of the independent and dependent variables can help students think about the relative rate of change of the variables of a linear function. Translation provides an excellent opportunity for connecting Geometric Functions to linear functions, since translation and addition are sliding operations in the plane and on the number line respectively. It's also important to connect the congruence of figures in this activity to students' previous experiences with congruence.

Launch

Expect to spend about 5 minutes.

Tell students that today they will explore the translation function family by creating and manipulating translation functions. They will then investigate the similarities and differences, and use the functions they create to make interesting shapes.

Explore

Expect students at computers to spend about 25 minutes.

Assign student pairs to computers and distribute the worksheet. Tell students that the less-experienced partner of each pair should operate the mouse first, and that the other partner should provide coaching and record observations without touching the mouse. Tell students to switch between their roles as operator and coach/recorder after they answer Q4 on the worksheet.

Tell pairs to agree on their answer to each question, and to record their answers in the sketch or on separate paper as you determine.

Circulate as students work, asking them what they notice as they drag the independent variable and observe the dependent variable. Encourage them to drag in various directions and at various speeds as they look for patterns in the relative rate of change of the variables. Make note of questions and difficulties that arise, and encourage students to raise these questions or difficulties during the concluding whole-class discussion.

Whenever possible ask questions instead of providing information. Pay particular attention to students' observations for Q1 ("What happens when you drag point x up?" or "Be sure to compare both the directions and the speeds.") and to their explanation for Q7 ("How does changing d or changing θ affect the relative rates and locations of the variables?"). As you check on the groups, plan the order in which to call on students to bring out aspects of function reasoning in a logical way. This can ensure that simple observations precede more sophisticated explanations and interpretations. Also, have selected pairs recall those questions or difficulties that would be beneficial for the entire class to discuss.

If time and technology permit, use a network folder or on a flash drive to collect several student sketches that illustrate interesting difficulties or interesting observations. As you collect sketches, consider the most logical order in which to show them. That way, you can lead a discussion about the relative rates of change of the variables and the shapes traced out when the domain is restricted.

Be sure to leave time for the class discussion and summary before the end of the period. If time is tight, consider making questions Q10 through Q12 optional.

Students who finish early should be encouraged to try **Translation Challenges.gsp.**

Discuss and Summarize

Expect to spend about 10 minutes.

Gather the class. Students should have their worksheets with them. Give students the opportunity to discuss difficulties or misconceptions.

Summarize the important concepts covered:

- varying the independent variable to observe the behavior of the dependent variable;
- labeling the dependent variables with notation that describes their relationship;
- attending to the relative motion of the two variables and to their fixed point(s);
- restricting the domain of the independent variable;
- tracing out the corresponding range; and
- comparing the shape of the restricted domain to the shape of the range.

Ask students to describe in their own words how they might distinguish between congruent shapes produced by reflection, rotation, and translation.

Based on the order you determined while circulating, call on students to discuss their answers for Q7. Because this is the last of the four transformation families students are investigating, ask them to compare and contrast these four families. What similarities did they find, and what differences? Encourage them to think about both the relative rate of change of the variables (covariation) and the relative shapes of the traces (correspondence). Ask students to describe in their own words how they might distinguish between congruent shapes produced by reflection, rotation, and translation. This discussion of relative shapes can help students form a comprehensive understanding of congruence that encompasses differences in location (translation), angle (rotation), and orientation or polarity (reflection).

The discussion can also touch on the connection between translation functions and linear functions. You could ask students to give an example of a linear function (expressed as $y = mx + b$) in which the values of x and y always change at exactly the same rate, just as the two variables of a translation function always move at exactly the same speed and direction. Elicit an example for which the value of y is always a certain amount greater than the value of x, and another for which the value of y is always a certain amount less than the value of x. You can ask students what type of translation function each of these cases corresponds to. There's no single correct answer; some students may think of "greater than" as a rightward translation and "less than" as leftward (by analogy to the number line), and others may think of "greater than" as upward and "less than" as downward.

Through these questions, and similar questions related to dilations, students can connect the value of b in $y = mx + b$ with a translation vector and the value of m with a scale factor. Students can begin to understand that addition is a sliding operation and multiplication is a scaling operation on the number line, just as translation is a sliding operation and dilation is a scaling operation on the plane.

Throughout the discussion, validate both correct and incorrect efforts by emphasizing the value of making mistakes, contemplating them, and persevering in order to learn from them.

Assess

Just before students leave, ask them to fill out an exit ticket describing one important thing they learned and one thing that they're not sure about.

Class discussion during the subsequent review, along with student work on the Translation Challenges, are important opportunities for assessment.

Review and Challenges

It's important to devote part of another class period to reviewing what students have learned so far and to consolidating that learning by means of the Translation Challenges. (It's best to do this during class time rather than for homework so that you can monitor and discuss students' work and assess their understanding.)

Expect to spend about 30 minutes.

Project the sketch **Translation Challenges Review.gsp**. Pages 1 through 4 list the various objectives of this activity, to facilitate a brief review that will help students' retention. On each page have a different student volunteer press the bullet(s) to reveal activity objectives, and then use the page to illustrate that page's objectives.
Page 1: As the student reveals the bullets, have a different student read the text beside each bullet point. When appropriate, have the student at the computer drag the independent variable of one or the other of the two functions on the right.
Page 2: Have a student read the text beside each bullet as it is revealed. These bullets list various characteristics of translation functions; have the student at the computer illustrate each of them by dragging an independent variable.
Page 3: The second, third, and fourth bullets describe three ways to define a translation function. As each bullet is read, have the student at the computer define a translation using the method described by the bullet. For instance, the second bullet describes a translation defined by a distance and an angle, so have the student translate point x by the distance parameter (d) and angle parameter (θ) shown next to the point.
Page 4: Create a visual picture of this function's operation by restricting the independent variable to the polygon (by selecting point x and the polygon, and then choosing **Edit | Merge Point to Polygon**). Then drag or animate x.

Send students to the computers in pairs, have them open **Translation Challenges.gsp,** and ask them to solve as many of the puzzles as they can.

One outcome of working on the puzzles is the ability to look at two translated shapes and visualize the vector in two-point form, in polar form, or in rectangular form.

[In a later activity, *Compose a Locus,* students will learn how to construct the range corresponding to a restricted domain. This technique makes it easier to solve puzzles such as these, but would be counter-productive if introduced now.]

Challenge 6 asks students to find the distance and angle that translates *M* to *N*, and then to find the distance and angle that translates *N* to *M*. (Note that there are solutions in which only the angle is changed, a solution in which only the distance is change, and solutions in which both are changed.) Thus the challenge is to find a function that will take the range back to the corresponding domain—in other words, to find the inverse function. (Don't introduce the terms *inverse* or *inverse function* yet. Each transformation challenge activity has a similar question; after answering these questions for several different function families, students will be ready to formalize and name the concept.)

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 Students should observe that x and $T_{4,60}(x)$ always move in the same direction and at the same speed. They may also observe that they are always the same distance apart, and always at the same angle.

Q2 Answers will vary, but should mention that the two sets of traces are the same shape and size. (Ideally students will use the term *congruence* in their description.) The traces are also oriented the same way.

Q3 There are no fixed points; the independent and dependent variables are always the same distance apart. A fixed point occurs when the distance between input and output is zero, which can never happen unless the translation function uses a distance of 0 units.

Q4 The independent variable is described as an independent object. The dependent variable is described as the translation of point x by 4 cm. at 60°.

Q5 Drawings will vary, but the two sets of traces should always be congruent. The dependent variable, $T_{d,\theta}(x)$, always moves at the same speed and in the same direction as x.

Q6 Drawings will vary. The shapes will be different, and either the distance or the angle will be different, depending on which parameter(s) were changed.

Q7 The shapes shown require an angle of approximately $210° \pm n \cdot 360°$ for integer values of n. (For example, if $n = -1$, the angle would be $-150°$.) Alternatively, the angle could be $30° \pm n \cdot 360°$ with a negative distance. Any value for the distance is acceptable; there is no way to determine distances from the picture.

Q8 The independent variable is restricted to the border of the polygon.

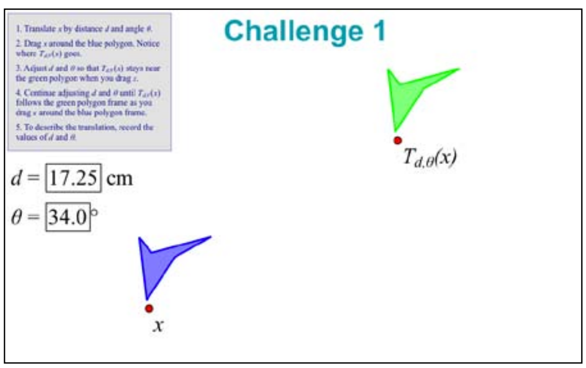
Q9 The range has exactly the same shape, size, and orientation as the restricted domain. The domain and range are congruent.

Q10 The variables move at exactly the same speed, in exactly the same direction.

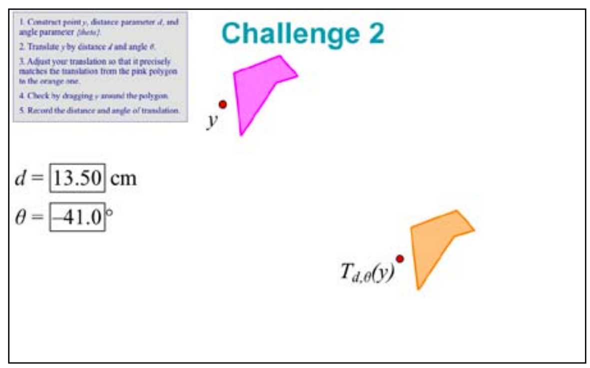
Q11 A translation cannot have fixed points, because the two variables are always separated by a constant distance. (The only exception occurs if the translation function is a shift of 0 units, in which case every location on the entire plane is a fixed point.)

Q12 Answers will vary. You can measure the distance between two corresponding points of the given shapes, and you can measure the angle between the horizontal and a line through the two corresponding points.

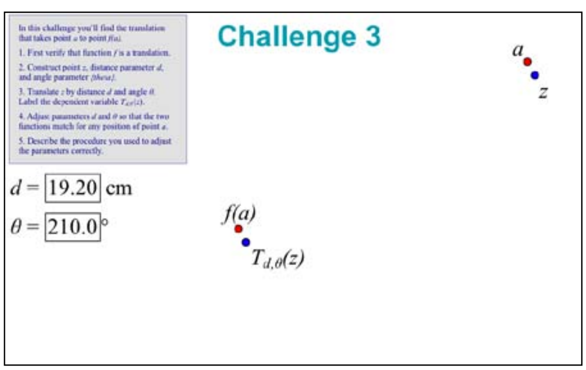
Translation Challenges



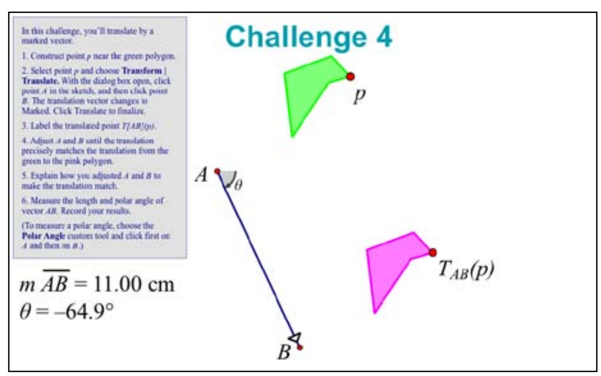
The translation is approximately 17.25 cm at 34°.



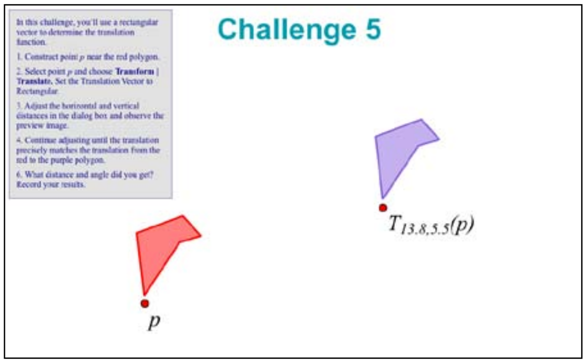
The translation is approximately 13.5 cm at –41°.



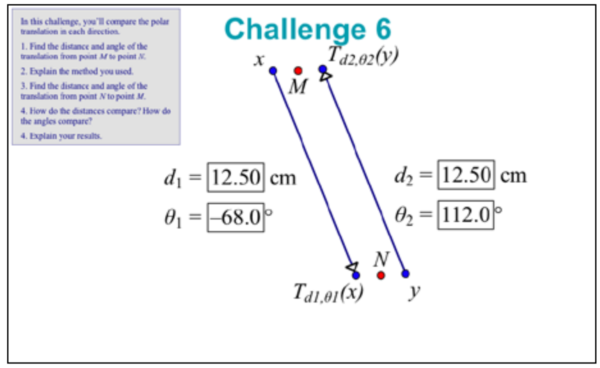
The translations match when the angle is 210° and the distance is 19.2 cm.



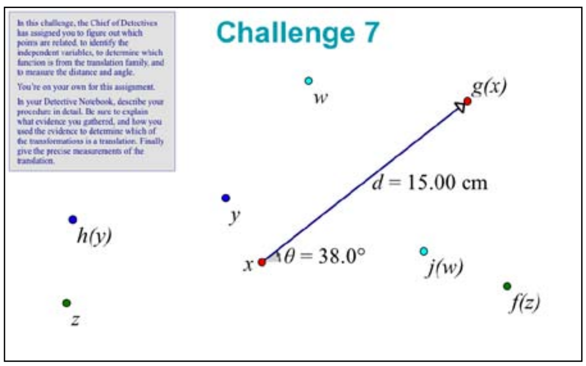
The translation is approximately 11 cm at –64.9°.



Independent point p was translated by 13.8 cm horizontally and 55 cm vertically.



The polar angle from M to N is –68°, and the polar angle from N to M is 112°, the exact opposite direction. This challenge suggests the concept of inverse functions, which will be addressed in a later activity.



Function g is the only translation on this page.

Related Activities:

- *ID the Suspects—Identify Functions (Recommended)*
- *Family Resemblances—Identify Function Families (Recommended)*
- *Reflection Challenges—The Reflection Family (Prerequisite)*
- *Rotation Challenges—The Rotation Family (Recommended)*
- *Dilation Challenges—The Dilation Family*
- *Family Relationships—Rotation, Translation, and Translation*
- *Dance the Dependent Variable—Geometric Function Dances*

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