Notes for the Teacher

This activity uses geometric points as independent and dependent variables to introduce function composition. Students define two functions and compose them by merging the independent variable of the second function to the dependent variable of the first.

This activity is designed to be students' formal introduction to the concept of function composition, and to help students avoid common misconceptions about composition. For students who are already familiar with composition of functions, this activity is a useful way to review the concept using a completely different representation, to challenge possible misconceptions, to spur students' thinking about the similarities between these two ways of exploring functions, and to stimulate them to generalize and move toward a more abstract understanding of the concept.

Any student, no matter their background, can benefit from constructing the functions, dragging their variables, observing the behavior of each function, merging the input of one to the output of the other, and using function notation to express each function and the composition of the functions. These direct sensory-motor experiences, and the conceptual metaphors to which they give rise, can help students develop a strong and coherent abstract understanding of function.

> This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Expect this activity to take a single class period (about 45 minutes).

Objectives:

In this activity students will:

- Construct two functions, name them using function notation, vary the independent variables, and observe and describe the behavior of the dependent variables.
- Merge the input of one function to the output of the other.

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[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Describe the composed function using function notation.
- Describe the relative rate of change of the variables of the composed function by direct observation, and explain the result in terms of the original functions.
- For a given domain, predict the range (including its location, orientation, shape and size) and identify the range as being similar and/or congruent to the domain.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

Before undertaking this activity, students should have done one or the other of the following:

- Three of the four function challenge activities (*Reflection Challenges, Rotation Challenges, Dilation Challenges, Translation Challenges), or*
- *Family Relationships—Rotation, Dilation, and Translation Families.*

These prior activities are also highly recommended:

- *ID the Suspects—Identify Functions*
- *Family Resemblances—Identify Function Families*
- *Dance the Dependent Variable—Geometric Function Dances*

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together in developing students' conceptual understanding of functions.

High Cognitive Demand: This activity provides tasks, particularly the Explore More task, for which there is no cut-and-dried procedure for students to follow. Particularly on the short form of the worksheet, both the directions and questions require student initiative, experimentation, and analysis.

Mathematical Habits of Mind, Reasoning and Sense Making: Students build the mathematical objects themselves and then explore and analyze their behavior. The short form worksheet encourages students to be self-reliant.

Inquiry: The worksheet contains probing questions that require students to manipulate, observe, and analyze. Students have to experiment to see what happens when they combine two functions, using the output of one as the input to another. The Explore More question expects students to create their own experiment and report on the results.

Cooperative Learning: Students work in pairs throughout the activity. Expect students to work purposefully in groups, to coach each other, and to discuss every part of the activity with their partners.

Assessment: You should assess students' understanding by visiting and questioning student pairs, not only observing their work but also encouraging and guiding them. Use the summary discussion to probe for students' understandings and confusions. The last page of the worksheet is an exit ticket to help you assess students' progress.

Differentiation: Different levels of students are supported by the worksheet, which is available in a long form (containing detailed Sketchpad instructions) and a short form (containing only a mathematical overview of the objects to be constructed). Student pairs work together, coaching each other and consulting with other pairs for additional support. The optional Explore More question is designed to engage students with different levels of background knowledge and to encourage self-directed work.

Questioning and Discourse: Since most discourse will take place between and among pairs during inquiry, it's important to encourage students to describe and explain their construction methods and their observations to each other. Use the summary discussion to focus students' thinking on the big ideas, and the role that mathematical ideas play in the activity. These concepts include variables, functions, domain, range, and relative rate of change.

Instructional Strategies: By varying the independent variable, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

Prepare by printing enough copies of the worksheet (**Transform Twice Worksheet.pdf**) for your class. Note that the worksheet is available in long and short forms; the forms contain the same questions, but the long-form worksheet contains more detailed instructions leading up to the questions. The short form is on a higher cognitive level, and gives students more responsibility for figuring out the details. It concentrates on the mathematical objects to be constructed and manipulated, omitting details of constructions students have encountered in previous activities. Students who have already constructed and investigated functions from several families should be ready to use the short form, though they may need to use the Help menu (**Help | Reference Center**) to remind themselves of specific techniques. Consider printing copies of both forms, making the long form available to students on an as-needed basis.

Review the two sketches that accompany the activity. Thought students do their work in a new sketch, the prepared sketches can be useful for other purposes:

Transform Twice Work.gsp shows a typical construction at various stages identified by the numbered steps of the long form of the worksheet, and may be useful when you discuss and summarize with the class.

Transform Twice Challenges.gsp presents three challenges involving function composition that are beyond the scope of the activity itself. These challenges can be used as extra credit or as a follow-up activity for an advanced class.

Launch

Expect to spend about 5 minutes.

Tell students that they will investigate *composition* of functions, and that the word *composition* comes from two roots: *com* (meaning *together*, as in *committee* or *combine*) and *pose* (meaning to put, as in *propose*, or in *pose a question*). In music, a *composition* is created by putting notes together; in English class, you write a *composition* by putting words together; in chemistry, table salt is *composed* of sodium and chloride ions; and in today's activity, you will *compose* two functions by putting them together.

Tell students that they'll first construct two separate functions (a dilation function and a translation function), and then *compose* them to investigate the new function created from the two original functions. Remind them to keep in mind these big ideas during the activity: how *variables* behave when they are dragged, the *relative rate of change* of the variables, how *function families* differ from each other, how and why to use *function notation, restricting the domain* to compare the shape and size of the *domain and range*, and *similarity and/or congruence* of the resulting shapes.

Pass out the worksheets to the class. If you're using the short form of the worksheet, tell students that they will be expected to remember the construction details from previous activities and by consulting with their partners and with other pairs, and to use the **Help | Reference Center** command as an additional support. Remind students that the each pair will have an operator (using the keyboard and mouse) and a coach/recorder (making suggestions and recording notes). Tell them to switch roles after they answer Q5.

Tell students they will be asked in Q8 to predict the behavior of a combination of the two functions. Emphasize the importance of predicting before dragging, as a check on their understanding and as a way of selecting what features to focus their attention on.

Explore

Expect to spend about 30 minutes.

Circulate as students work, making sure students write clear descriptions in complete sentences of the behavior they observe. Emphasize the importance of noting relative rate of change, fixed points, and other features.

For students using the short form, the most accessible form of assistance is the diagram on the worksheet. By showing a graphic example of how the finished construction should look, it provides help in a condensed and mathematically-meaningful way.

Note that Q2 and Q4 refer to functions as objects: $D_{C,2}$ and T_{DE} . This is a new usage, but one which students should be prepared for based on their earlier experiences. Pay close attention to students' answers, and as you circulate, consider asking what they think this notation means.

Pairs that finish early should do the Explore More: constructing a single function that has the same effect as the composed function $T_{DE} \circ D_{C,2}$. This new notation using the ring symbol is described on the worksheet; ask students about it as you circulate.

Instead of the Explore More, you can challenge students to create other compositions. For instance you could suggest that they compose two translations, or two rotations, or a dilation by 2 and a dilation by ½, or a function with itself. You might encourage such students by challenging them to create the most interesting composition they can.

Be sure to call students to a halt early enough to conduct a summary discussion.

Discuss and Summarize

Expect to spend about 10 minutes.

Composition is a topic that's often confusing to students; one objective of this activity is to provide a simple, concrete example, created by the students themselves, to ground their thinking about this challenging concept. Accordingly, the class discussion and summary are a critical element of this activity.

Gather the class and spend a minute or two asking students for feedback: what did they learn, what was easy, what was difficult, and what was interesting?

If you need to refer to a completed sketch during the summary discussion, you can use a student sketch that you've collected. Alternatively, you can use **Transform Twice Work.gsp**

Below are some issues and questions that should be addressed.

- Ask students to comment on how the relative rates of change of the variables distinguish one function from another. By questioning students about observed covariation, they'll become conscious of function behavior and of the importance of the way the variables vary. They'll also be better prepared to discuss the slope of linear functions and the shape of a quadratic graph as indications of the relative rate of change of the independent and dependent variables.

- Students should become fluent in reading and understanding function notation. By attaching meaning to the notation, many misconceptions can be avoided or, if already present, ameliorated. For example, the common misreading of $f(x)$ as "f times x" is easier to prevent or correct when the symbol for the function is an abbreviation for a transformation such as dilation (D) or translation (T).

- Review how students relabeled the final dependent variable from $T_{DE}(w)$ to $T_{DE}(D_{C,2}(x))$. This step is critical to their ability to meaningfully read the resulting function notation.

- Students should note that reading the label $T_{DE}(D_{C,2}(x))$ from left to right actually names the composed functions in the reverse of the order in which they were applied. Challenge students to make sense of this. One possible explanation is that the label names the last point (the dependent variable), so the description has to start from the end of the process. That is, the final variable is a translated image (of the intermediate variable), and the intermediate variable is a dilated image. A related explanation refers back to the algebra principle that the innermost parentheses must be evaluated first, so this expression indicates that the process begins with the innermost part (x) and proceeds outward.

- The activity encourages students to read $D_{C,2}(x)$ as "the dilation, about C..." rather than "the dilated image, about C..." This choice is intentional but possibly confusing, because "dilation" actually refers to the function rather than to the point. The important thing is to encourage students to think of $D_{C,2}$ as the name of the function and $D_{C,2}(x)$ as the name of the variable that results from applying the function $D_{C,2}$ to independent variable x. The use of T_{DE} and $D_{C,2}$ to refer to the functions themselves is emphasized in Q2, Q4, Q10, and Q12.

- Call attention to the notation $T_{DE} \circ D_{C,2}$ introduced in the Explore More section. Ask students why this notation might be useful. Isn't $T_{DE}(D_{C,2}(x))$ good enough for expressing composition? You may have to probe a bit until students realize that $T_{DE}(D_{C,2}(x))$ names the dependent variable, but doesn't include a name for the composed function itself.

- Explore More question Q12 is an interesting challenge for students. It asks them to take a closer look at the composed function, with the middle variable hidden, and decide whether it belongs to a family they already know. If they decide it's a dilation, then the challenge becomes figuring out whether or not it's the same dilation as $D_{C,2}$. If they decide it's different, they must determine its center and how it is related to point *C*. This challenge could lead to investigations of how other compositions of functions may turn out to belong to a known family. For instance, if the composition of two translations is another translation, what is its vector? It can also lead students to try to create a composition that does not belong to one of the known families; they could discover glide reflections through such an exploration.

A possible final question for students is this: How do you think compositions might be useful in the real world? Where can you find examples of them?

Consider providing students with the sketch **Transform Twice Challenges.gsp** as an extra-credit assignment, or alternatively just for fun.

Assess

Just before class ends, ask students to fill out the exit ticket. Question 1 can provide useful information about students' understanding of the main topic, and question 2 is intended both to help assess students' understanding of function notation and to help prevent some of the notational confusion that often results when students learn about composition through algebraic abstractions.

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 The drawing should show the labeled center point and the paths traced by the two variables. The dependent variable's path should be a dilated image similar to, but twice as large as, the independent variable's path.

Q2 Possible observations include (a) $D_{C,2}(x)$ moves faster than *x*, (b) $D_{C,2}(x)$ stays twice as far from center point *C* as *x* does, (c) the trace of $D_{C,2}(x)$ is twice as large as the trace of *x*, and (d) the only place the two variables meet is point *C*. Other answers are possible.

Q3 Drawings will vary, but the two traces should be congruent and displaced by the distance and direction of the vector.

Q4 Possible observations include (a) $T_{DE}(w)$ moves at the same speed and in the same direction as w, (b) $T_{DE}(w)$ always stays the same distance and direction from w, (c) the trace of $T_{DE}(w)$ is the same shape and size as the trace of w, and (d) the variables never meet, because they're always separated by vector *DE*. Other answers are possible.

Q5 Point w moved to point $D_{C,2}(x)$ and combined with it.

Q6 You can read the label $T_{DE}(D_{C,2}(x))$ as "the translation, from *D* to *E*, of the dilation, about *C* by a scale of 2, of independent variable *x*." This makes sense because it describes how you can find the dependent variable starting with the independent variable.

Q7 Drawings will vary. The intermediate trace should be twice the size of the independent variable's trace, and the dependent variable's trace should be the same size as the intermediate trace, but translated.

Q8 The predicted range should be the same shape as the original polygon domain, but twice its size and translated. The most important thing about the prediction is that it be done before dragging, regardless of its accuracy.

Q9 Drawings will vary, but should show a range twice the size of the original polygon and then translated.

Q10 Descriptions will vary. Function $D_{C,2}$ connects the domain points to the traces of the intermediate variable, and function T_{DE} connects the traces of the intermediate variable to the traces of the dependent variable (the points of the range).

Q11 The dependent variable $T_{DE}(D_{C,2}(x))$ moves twice as fast as the independent variable *x*, and in the same direction. The fixed point seems to be center point *C* translated by the reverse of the vector *DE*.

Q12 The crux of Q12 is to actually construct the single dilation function that is equivalent to the composition $T_{DE} \circ D_{C,2}$. The solution is a dilation with a scale factor of 2 and with its center point at the translated image of point *C* by the reverse of translation vector *DE*. (In other words, translate *C* by vector *ED* to construct the center point of the composed function.)

Related Activities:

- *ID the Suspects—Identify Functions (Recommended)*

- *Family Resemblances—Identify Function Families (Recommended)*
- *Reflection Challenges—The Reflection Family (Prerequisite)*
- *Rotation Challenges—The Rotation Family (Recommended)*
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Translation, and Translation Families*

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