Notes for the Teacher

In this activity students go beyond the standard geometric functions built into Sketchpad to create a new function family that creates a swirling effect. The swirl provides fresh ground for investigating function behavior and the relationship between domain and range, because it behaves differently from any of the function families students have previously explored.

In addition to its value as incentive for students to revisit important function concepts in a new light, this activity serves as a lead-in for its companion activity, *Animated Special Effects—Swirl a Picture*. Though easily constructed, the swirl is a sufficiently interesting and complex function that it can form the basis of a very impressive visual special effect when applied to a picture.

Like the prerequisite activity *Compose a Locus—Composition, Domain, and Range*, this activity continues with an emphasis on functions as mappings and functions as objects. As a mapping, a function acts on an entire domain all at once to produce the corresponding range. As an object, a function is subject to being acted upon, and can itself be transformed into other members of its function family.

Expect the activity to take a single class period (about 45 minutes). A second class period could be profitably spent having students create special effects of their own, as prompted by the Challenge document.

> This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Create a rotation function using a fixed parameter as the angle.
- Redefine the function definition to make the angle of rotation depend on the location of the independent variable.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Describe the covariation behavior of the modified function.
- Investigate the function as a mapping by restricting its domain, constructing the corresponding range, and comparing their shapes.
- Predict the behavior of other members of the same family, and change the function to test the predictions.
- Animate the function to explore a continuum of members of the function family, and explain the resulting “special effect” in terms of the function that produces it.
- Conduct further investigation to quantify the attributes of the swirl in terms of the relative rates of change of the variables and the shapes of the restricted domain and corresponding range.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

Before undertaking this activity, students must already be familiar with constructing a locus to embody the range of a function, by doing the following activity:

- *Compose a Locus—Composition, Domain, and Range*

Students should also have completed one or the other of the following:

- Three of the four function challenge activities (*Reflection Challenges, Rotation Challenges, Dilation Challenges, Translation Challenges), or*
- *Family Relationships—Rotation, Dilation, and Translation Families.*

These prior activities are also highly recommended:

- *ID the Suspects—Identify Functions*
- *Family Resemblances—Identify Function Families*
- *Dance the Dependent Variable—Geometric Function Dances*

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together to develop students' conceptual understanding of functions.

Cognitive Demand: The short form of this activity contains high level directions and questions that require student initiative, experimentation, and analysis. The questions on both forms connect students' sensory-motor experiences to the mathematical objects they are manipulating and observing.

Mathematical Habits of Mind, Reasoning and Sense Making: Students build all of the mathematical objects and then explore and analyze the behavior of these objects. The short form worksheet encourages students to be particularly self-reliant. Students are encouraged to make mathematical sense of the animation of parameter k by thinking of the function as a mapping from the domain to the range.

Inquiry: The worksheet contains probing questions that require students to manipulate, observe, and analyze. As students work, they will pose and answer questions, such as: Why does the independent variable behave the way it does? How will different values of parameter k affect the function? With the independent variable restricted to a ray, how will the shape of the range change when I animate parameter k? The Explore More question expects students to create their own experiments and report on the results.

Cooperative Learning: Students work in pairs throughout the activity. Expect students to work purposefully together, to coach each other, and to discuss every part of the activity with their partners.

Assessment: You should assess student understanding by visiting and questioning student pairs, not only observing their work but also encouraging and guiding them. Use probing questions to encourage students to assess their own work and their own explanations. Use the summary discussion to probe students' depth of understanding and possible misconceptions. The last page of the worksheet is an exit ticket.

Differentiation: Different levels of students are supported by the worksheet, which is available in a long form (containing detailed Sketchpad instructions) and short form (containing only a mathematical overview of the objects to be constructed). Student pairs work together, coaching each other and consulting with other pairs for additional support. The optional Explore More questions are designed to engage students with different levels of background knowledge and to encourage self-directed work.

Questioning and Discourse: Since most discourse will take place between and among pairs during inquiry, it's important to encourage students to describe and explain their construction methods and their observations to each other. Use the summary discussion to focus students' thinking on the big ideas, and the role that mathematical ideas play in the activity. These concepts include functions as mappings from domain to range, and functions as objects that can themselves be transformed through a continuum of members of a function family.

Instructional Strategies: By varying the independent variable, and later animating parameters to transform functions themselves through a set of related family members, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

Prepare by printing copies of the worksheet (**Special Effects Worksheet.pdf**) for your class. Note that the worksheet is available in long and short forms; the forms contain the same questions, but the long form contains more detailed instructions leading up to the questions. The short form is on a higher cognitive level, and gives students more responsibility for figuring out the details. It concentrates on the mathematical objects to be constructed and manipulated, omitting many of the simpler construction details. Most students who have completed the prerequisite activities should be ready to use the short form, though they may need to use the Sketchpad Tips (**Help | Using Sketchpad**) or the Reference Center (**Help | Reference Center**) to remind themselves of some techniques. Consider passing out the short form to all the teams and printing a few copies of the long form for students to consult as needed.

The worksheet pdf also includes a two-page answer sheet that lists all the questions from the worksheet with space for students to write their answers. An exit ticket is also included that you can pass out near the end of class to provide you with feedback while provoking students to think about what they've learned.

Review the two sketches that accompany the activity. Students do their work in a new sketch, but these sketches can be useful for other purposes. **Special Effects Work.gsp** shows a typical construction at various stages identified by the numbered steps of the long form of the worksheet. **Special Effects Challenges.gsp** presents four challenges that are beyond the scope of the activity itself but can serve as extra credit or as a follow-up activity for an advanced class.

Launch

Expect to spend about 5 minutes.

Explain to students that they will start by creating a function they know, and then modify its behavior to make a very different function. By creating this new function, they'll realize they've actually created a new function family. By animating the function to turn it into different members of its function family, they'll create interesting special effects.

Tell students that they already know most of what they need for the construction of the swirl function. For instance, they already know how to use the **Locus** command to construct the range corresponding to a restricted domain. They also know how to use **Edit | Action Buttons | Animation** to make a button that animates a parameter. Mention these commands specifically to help students recall this necessary information. One new command they will have to learn to use is **Edit | Edit Parameter**, which they can use to turn a parameter into a more complicated calculation.

Tell students they will be continuing to observe covariation (how two variables change in relation to each other) and correspondence (which is easiest to see by comparing a restricted domain with its corresponding range). Finally, tell them that the special visual effect they're going to create will appear when they animate their function in Q11, at the end of the regular part of the worksheet.

Pass out the worksheets to the class. If you use the short form of the worksheet, tell students that they will be expected to figure out the construction details themselves by experimenting, by consulting with their partners and classmates, or by using a copy of the long form that will be made available to them. Students should use the Sketchpad Tips or the Reference Center for additional support. Remind students that each pair will have an operator (using the keyboard and mouse) and a coach (making suggestions and recording notes). Tell them to switch roles right after they answer Q4.

Remind students to monitor their own understanding of function concepts, and to keep track of what they understand well and what they're unsure about.

Explore

Expect to spend about 30 minutes.

Circulate as students work, making sure they write clear descriptions in complete sentences of the behavior they observe. Emphasize the importance of noting relative rate of change, fixed points, and other features.

Be prepared for problems students might have with editing parameter θ (in step 6 of the long form or step 3 of the short form), as they must create a new parameter k while editing θ. By editing parameter θ, students modify the function and turn it into a member of a new function family. Questions Q7 through Q11 are also challenging, as students vary and animate k to create different members of their new family.

In the Explore More, students will quantify the relative rate of change of the variables for given values of the independent variable, and take a more detailed look at the correspondence between domain and range.

If you have students reach the last question (Q15 in the Explore More), it may be more interesting to those students to skip Q15 and instead go direction to the Challenge sketch.

Be sure to call students' work to a halt early enough to conduct a summary discussion.

Summarize

Expect to spend about 10 minutes.

Ask students, "After you edited the function in step 3 (short form) or step 6 (long form), what can you say about how the dependent point is related to the independent point?" Students should identify the independent point as the one that's being rotated, so it serves as the starting location for rotation. Students should also observe that after editing, the value of θ now depends on the independent variable, because measurement *Cx* involves the location of *x*.

Ask students to explain what they saw when they pressed the action button in Q11. After several descriptions, ask them what changed mathematically to produce the visual effect of a spiral alternately winding and then unwinding. Their explanations are likely to involve mathematical details rather than the bigger picture. You can challenge this detail-oriented view by pointing out that the domain isn't changing and the independent variable isn't moving, so how can the range be changing? What about the mathematics of the function is changing? Students should realize that the function itself is changing as it continually becomes different members of the same function family. In fact, every possible value of *k* corresponds to a different member of the family, and produces a different shape of the range.

Students' early experiences have been of a function as an action or procedure that can be applied to an independent variable to produce a dependent variable. But in this activity they experience a function as an object that can be varied over time. This is an important (and difficult) conceptual step for students; don't expect full understanding through this one discussion.

During the summary discussion, you can use as an example either a student sketch that you've collected or **Special Effects Work.gsp**. Either sketch can be useful for reviewing steps or for answering student questions about their investigation.

The Big Ideas page of **Special Effects Work.gsp** contains bullet points summarizing some important ideas from this activity, and may be a useful way to end the discussion.

- *I can edit a simple function to make a more complex one.*
- *The relative rate of change determines the function's behavior (covariation).*
- *I can use the mapping from domain to range to understand and control a function's behavior (correspondence).*

- *By inventing one new function, I can invent a whole family.*
- *A function is not just a procedure; it's also an object that I can control and change in interesting ways.*
- *I can animate a function to produce many family members and display an interesting "special effect."*

Consider providing students with the sketch **Special Effects Challenges.gsp** as a fun challenge, or as an extra-credit assignment. Sample solutions for the first three challenges are contained in **Special Effects Work.gsp.**

Assess

Before class ends, ask students to fill out the exit ticket. Their answers can help you evaluate what students took away from the activity. Tell students the exit ticket can also be used for students to write their own questions or to describe their own difficulties regarding the activity.

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 The dependent variable moves at the same speed as the independent variable, but in a direction that differs by the angle of rotation, 45°. There's a single fixed point at the center of rotation, point *C*. (This question and the next are review questions to remind students of work they've already done.)

Q2 When θ is 15°, the difference in the direction of motion of the two variables is only 15°, so they move in nearly the same direction.

Q3 Predictions will vary. (If the calculation is done correctly, the value of θ should be 45° when Cx = 3 cm, and 90° when Cx = 6 cm. Actual results will differ slightly, because it's hard to position x so it's at an exact distance from *C*.)

Q4 Students should report on the accuracy of their predictions. They may describe the motion of the dependent variable, such as the observation that dragging the independent variable in a straight line causes the dependent variable to follow a curve.

Q5 The range is now a curve rather than a straight line. It curves farther away from the ray as x is dragged farther away from *C*.

Q6 Pictures will vary. The explanation should include the fact that the angle of rotation increases the farther *x* is from *C*.

Q7 Values students choose for *k* will vary, as will their predictions. Students should propose a specific value for *k*, and make a specific prediction for its effect. Students may predict that values of *k* larger than 15° will make the curve go farther away from the ray, while values less than 15° will make it stay closer to the ray. Students are unlikely to predict that the range will ever loop back around to cross the domain.

Q8 Drawings and descriptions will vary. Students should include an assessment of the accuracy of their prediction.

Q9 Predictions will vary. Testing the prediction will result in the curve moving to the opposite (clockwise relative to point *C*) side of the ray.

Q10 Even for negative values of *k*, most previous observations hold; the range curves away from the ray, and more strongly for values of *k* with larger absolute values. Students are likely to describe only a single fixed point at *C*, though some may have tried a large enough value for *k* to have observed additional fixed points.

Q11 Students will observe that the range is a curve winding and unwinding around *C* until it crosses the ray, when it begins winding and unwinding in the opposite direction. The animation of *k* between 90° and –90° provides large enough values of *k* that students will observe the range crossing over the domain, with the first such intersection occurring 4 cm from point *C*.

Q12 The fixed points occur when θ is a multiple of 360°. When $k = 120°$, the fixed points occur when *x* is at *C*, when *x* is 3 cm from *C*, when it's 6 cm from *C*, and so forth.

Q13 *W*hen *x* is 6 cm from *C*, the dependent variable is moving nearly perpendicularly to the ray. The dependent variable moves at a speed that will bring it back to the ray by the time *x* has moved 3 cm more outward. Thus the dependent variable will move around 1/3 of a circle in the time *x* moves 1 cm. The ratio of the speeds of the two variables will be 1/3 the circumference of a circle of radius 6, or 4π. So the dependent variable is moving more than 12 times as fast as *x*. (This question requires students to think about the instantaneous rate of change of the dependent variable with respect to the independent variable. Thinking about such questions now will help to prepare students for a later course in calculus.)

Q14 Drawings and explanations will vary. Students should explain that this function must map straight-line domains to curved ranges, because different points on a straight line cannot be the same distance from *C*. As a result, the angle of rotation changes as the independent variable moves along the domain. Careful consideration of just one side of the polygon reveal how the corresponding portion of the range must curve based on what part of the side is closest to *C* and what part is farthest from *C*.

Q15 Students' invented functions will vary. Note that the challenges in **Special Effects Challenges.gsp** provide more scaffolding to help students invent and animate interesting new functions.

Related Activities:

- *ID the Suspects—Identify Functions*
- *Family Resemblances—Identify Function Families*
- *Reflection Challenges—The Reflection Family*
- *Rotation Challenges—The Rotation Family*
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Translation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*
- *Transform Twice—Function Composition*
- *Compose a Locus—Composition, Domain, and Range*
- *Animated Special Effects—Swirl a Picture*

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Portions of this material are based upon work supported by the National Science Foundation under award number DRL-0918733. Any opinions, findings, and conclusions or recommendations expressed in this work are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.