Notes for the Teacher

Students explore rotations as functions. They construct a rotation function, label the variables using function notation, and trace and compare the paths of the variables. They restrict the domain to reveal the corresponding range, and they change the angle and move the center to create different members of the rotation family. Finally, students solve rotation challenges by figuring out the center and angle to match an existing rotation.

Note that the material in this activity is also covered in the activity *Family Relationships—Rotation, Dilation, and Translation Families*. This activity contains more explicit student directions (following the pattern established in the prerequisite activity on the reflection family) and covers only the rotation family. The *Family Relationships* activity is more ambitious, expecting students to work independently as they explore and compare the rotation, dilation, and translation families. You should use that activity if your students are prepared for its more independent approach.

This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Construct an independent variable point *x*, a center point, an angle parameter, and the rotated image of *x* defined by the center and angle.
- Label the dependent variable using function notation.
- Drag the independent variable while tracing both variables, and describe both the variables' relative motion (their *covariation*) and the relationship between their traces (their *correspondence*).
- Change the angle to form and investigate a different member of the same family.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Restrict the domain of the independent variable to the border of a polygon, and observe and describe the resulting range by dragging the independent variable.
- Identify fixed points of the function, and explicitly compare the relative motion (both speed and direction) of the independent and dependent variables.
- Solve challenges that involve finding a hidden center and unknown angle, given an object and its rotated image.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

- *Reflection Challenges—The Reflection Family* (Prerequisite)
- *ID the Suspects—Identify Functions* (Recommended)
- *Family Resemblances—Identify Function Families* (Recommended)

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together in developing students' conceptual understanding of functions.

High Cognitive Demand: This activity provides several tasks for which there is no cut-and-dried procedure for students to follow. Though the worksheet provides fairly explicit directions to help students perform the initial construction, the questions it asks require experimentation, inquiry, and analysis.

Mathematical Habits of Mind, Reasoning and Sense Making: Students construct and investigate mathematical objects, and are challenged to answer questions that require tinkering and analysis to understand the behavior of these objects.

Inquiry: The body of the activity supports student inquiry, and the worksheet contains probing questions that require students to manipulate, observe, and analyze.

Cooperative Learning: Students work in pairs during the exploration portion of the activity, and exchange roles between driving (using the mouse and keyboard) and coaching/recording. Expect students to work purposefully in pairs, to coach each other, and to discuss every part of the activity with their partners. The members of each pair should share the construction and manipulation steps equally, so that each student is fully involved in both creating the function and working with it.

Assessment: You should engage in formative assessment by visiting and questioning student pairs during the exploration phase. You can use the summary discussion to elicit students' understandings, confusions, and questions. The last page of the worksheet is an exit ticket.

Differentiation: The worksheet comes in a long form (with more explicit instructions) and a short form (for students with more experience with Sketchpad). It also includes an optional Answer sheet on which students can write their answers. The concluding Rotation Challenges entail varying levels of difficulty, and are very useful for furthering students' understanding of the Rotation Family and of function concepts in general.

Questioning and Discourse: Most discourse will take place between team members during inquiry, so it's important to encourage team members to describe their observations to each other and to discuss their answers to the questions. The questions on the worksheet, along with teacher observations of the work of different teams, should guide the summary discussion. Since the worksheet includes explicit directions for completing the construction, it's important to ask questions that focus students' thinking on the big ideas, both while they are working in pairs and during the whole-class discussion.

Instructional Strategies: By varying *x*, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

The time students will require for this activity can vary significantly depending on their mathematical background and Sketchpad experience. Because the concluding class discussion is critical, you should be prepared to postpone the challenges contained in **Rotation Challenges.gsp** to another day. (You may even elect to make worksheet questions Q9 and Q10 optional.) While you can assign the Rotation Challenges as homework exercises, it's preferable to have students work cooperatively in class while you observe their investigations.

The three stages of the activity are described in more detail below, in the sections on Launch, Inquiry, and Summary.

Worksheet: The worksheet comes in two forms, both of which contain the same questions. The longer form is useful for students who are relatively new to Sketchpad, as it provides explicit steps to follow. If students have experience with Sketchpad, the shorter form is preferable, as it expects more initiative from students and it concentrates more on the mathematics to be explored and less on technicalities. (Giving students unnecessarily explicit directions risks miring students in details, discouraging their thoughtfulness and creativity.) Some teachers may choose to provide one short form and one long form to each pair of students, telling them to work from the short form, referring to the details in the long form as needed.

Answer Sheet: Many teachers prefer their students to write their answers on their own paper. Alternatively, you might ask students to submit their answers electronically, or you can use the provided answer sheet. If you assign electronic submission, tell students that they can include screen captures by choosing **Edit | Select All** and then **Edit | Copy** in Sketchpad. They can paste the resulting image into a word processor, email, or other document. (The image includes traces and is cropped to the window border, so that students can easily control the area being copied and pasted.)

Transfer of Learning: To get the most value from this activity, students must connect what they've learned to other representations of functions. By dragging the independent variable *x* in this activity, they can relate its smooth motion to the continuous variation of a numeric variable. Moreover, comparing the relative speed and direction of the independent and dependent variables can help students think about the relative rate of change of the variables of a linear function. If students are already familiar with functions in other contexts, you should mention the importance of making connections during the introduction of the lesson, and devote some portion of the summary discussion to these connections.

Launch

Expect to spend about 10 minutes.

To activate students' prior knowledge, start Sketchpad and display page 3 of **Family Resemblances.gsp.** (Students will remember this page from a previous activity. Three of its four functions belong to the rotation family.) Ask a student to drag independent variable *d* while classmates observe the behavior of *j*(*d*). Ask students for a short summary of how to recognize when functions come from the same family. (Two or three brief student answers suffice; detailed discussion or review is not helpful at this stage.)

Tell students that today they will explore the rotation function family by creating and manipulating similar functions. They will then investigate the similarities and differences, and use the functions they create to make interesting shapes.

Remind students to use the Help menu if they have difficulties, and tell them that in addition to the Reference Center they may find the Sketchpad Tip on Rotating and Dilating useful. (Choose **Help | Using Sketchpad | Sketchpad Tips | Transform | Rotating and Dilating.**)

Explore

Expect students at computers to spend about 25 minutes.

Assign student pairs to computers and distribute the worksheet. Tell students that the less-experienced partner of each pair should operate the mouse first, and that the other partner should provide coaching and record observations without touching the mouse. Tell students to switch between their roles as operator and coach/recorder after they answer Q4 on the worksheet.

Tell pairs to agree on their answer to each question, and to record their answers in the sketch or on separate paper as you determine.

Circulate as students work, asking them what they notice as they drag the independent variable and observe the dependent variable. Encourage them to drag in various directions and at various speeds as they look for patterns in the relative rate of change of the variables. Make note of questions and difficulties that arise, and encourage students to raise these questions or difficulties during the concluding whole-class discussion.

Whenever possible ask questions instead of providing information. Pay particular attention to students' observations for Q1 ("What happens when you drag point *x* up?" or "How would you describe the difference between the two directions?") and to their explanations for Q5 and Q8 ("How does the angle parameter relate to the relative direction of the variables? What would happen if $\theta = 10°$?"). As you check on the groups, plan the order in which to call on students to bring out aspects of function reasoning in a logical way. This can ensure that simple observations precede more sophisticated explanations and interpretations. Also, have selected pairs recall those questions or difficulties that would be beneficial for the entire class to discuss.

If time and technology permit, use a network folder or on a flash drive to collect several student sketches that illustrate interesting difficulties or interesting observations. As you collect sketches, consider the most logical order in which to show them. That way, you can lead a discussion about the relative rates of change of the variables and the shapes traced out when the domain is restricted.

Students who finish early should be encouraged to try **Rotation Challenges.gsp.**

Discuss and Summarize

Expect to spend about 10 minutes.

Gather the class. Students should have their worksheets with them. Give students the opportunity to discuss difficulties or misconceptions.

Summarize the important concepts covered:

- varying the independent variable to observe the behavior of the dependent variable;

- labeling the dependent variables with notation that describes their relationship;
- attending to the relative motion of the two variables and to their fixed point(s);
- restricting the domain of the independent variable;
- tracing out the corresponding range; and
- comparing the shape of the restricted domain to the shape of the range.

Link the discussion of the shapes to the concept of congruence; this is an opportunity for students to describe in their own words what it means for two figures to have exactly the same shape and size, and to relate the term *congruence* to their own observations.

Based on the order you determined while circulating, call on students to discuss their answers for Q1, Q5, and Q8, and to compare the answers to these three questions. Use these particular students' contributions to try to generate a lively and probing discussion, and through the discussion to develop a class consensus on the relationship between the rotation angle and the relative direction of motion of the variables.

For Q8, students will have observed that dragging x left causes the dependent variable to move right, and dragging x down causes the dependent variable to move up. Connect their answers to linear functions (expressed as $y = mx + b$) by asking what value of m would make x and y move in opposite directions (one increasing while the other decreases) but at the same speed. (You might also ask students what rotation angle would make the variables move in the same direction. This question can lead to another interesting discussion if students propose more than one answer.)

In preparation for the Rotation Challenges sketch, show page 1 of that sketch and ask students to think about how you could start with a polygon and its rotated image to find the hidden center point and angle of rotation. Don't allow students to express solution strategies at this time.

Assess

Just before students leave, ask them to fill out an exit ticket describing one important thing they learned and one thing that they're not sure about.

Class discussion during the subsequent review, along with student work on the Rotation Challenges, are important opportunities for assessment.

Review and Challenges

Consider devoting part of another class period to reviewing what students have learned so far and to consolidating that learning by means of the Rotation Challenges. (It's best to do this during class time rather than for homework so that you can monitor and discuss students' work and assess their understanding.)

Expect to spend about 30 minutes.

Project the sketch **Rotation Challenges Review.gsp**. Pages 1 through 4 list the various objectives of this activity, to facilitate a brief review that will help students' retention. On each page have a different student volunteer press the bullet(s) to reveal activity objectives, and then use the page to illustrate that page's objectives.
Page 1: Ask the student to press each phrase in italics to reveal in turn the independent variable, the center point, the angle parameter, the dependent variable, and the function notation. Then have the student drag point x while observing the relative motions of the variables.
Page 2: After showing the first bullet, have the student drag x first vertically and then horizontally while observing the relative rate of change for both motions. After the second bullet, ask the student to press the italicized phrase *Move the center*, creating a different member of the same function family. Have the student drag x again, and observe the relative motion of the dependent variable. Ask the student to press the italicized phrase *change angle* θ, creating yet another member of the same function family. Have the student drag x again, and observe the corresponding motion of the dependent variable.
Page 3: Show the first bullet and have the student drag x to find a fixed point, and to verify that there's only one fixed point. Show the second bullet and have the student merge the independent variable x to the polygon and then drag x to show the effect of a restricted domain.
Page 4: This page introduces the rotation challenges from **Rotation Challenges.gsp**. Explain that one of the challenges involves finding a hidden center and angle parameter that connects the red polygon to the blue one. Having a volunteer do these steps:

1. Drag the independent variable x.
2. Restrict x to the red polygon.
3. Erase traces and animate x.
4. Drag C to adjust the center, and edit θ to change the angle.
5. Erase traces and animate again.

Before the volunteer determines the center or angle correctly, stop the demonstration and tell students that discussing and solving these puzzles will develop their intuitive understanding of the rotation family. Send students to the computers in pairs, have them open **Rotation Challenges.gsp,** and ask them to solve as many of the puzzles as they can.

One outcome of working on the puzzles is the ability to look at two rotated shapes and quickly approximate the angle of rotation. Students' strategy is likely to involve mentally rotating the original polygon in place until its sides are parallel with the sides of the range polygon. Finding the center point through imaginary visualization is harder; don't worry if most students use guess-and-check methods to find the fixed point, by dragging the polygons to bring corresponding vertices together.

[In a later activity, *Compose a Locus,* students will learn how to construct the range corresponding to a restricted domain. This technique makes it easier to solve puzzles such as these, but would be counter-productive if introduced now.]

Challenge 6 and 7 merit particular attention during the concluding class discussion.

Challenge 6 makes the connection between rotation and congruence, and represents an opportunity to develop this connection by building on the discussion of congruence from the Reflection Challenge activity.

Challenge 7 has two polygons, P_1 and P_2 without identifying which is the restricted domain and which is the range. The challenge asks students to find the center and angle that rotates P_1 to P_2, and then to find the center and angle that rotates P_2 to P_1. Thus the challenge is to find a function that will take the range back to the corresponding domain—in other words, to find the inverse function. (Don't introduce the terms *inverse* or *inverse function* yet. Each transformation challenge activity has a similar question; after answering these questions for several different function families, students will be ready to formalize and name the concept.)

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 Students should observe that when they drag x up $R_{C,90}(x)$ moves left, and when they drag x left $R_{C,90}(x)$ moves down.

Q2 Answers will vary, and may include observations like these: The shapes look the same, but one is a rotated version from the other. Horizontal parts of one shape are vertical on the other. The dependent variable's shape is turned 90° counter-clockwise from the independent variable's shape. Both shapes are the same size. Corresponding points of both shapes are the same distance from point *C*.

Q3 When you go through the center, the independent and dependent variables meet. This happens because when you rotate a point that's already at the center, it can't go anywhere. This is the only fixed point.

Q4 The independent variable is described as an independent object. The dependent variable is described as the rotation of point x by 90° about center point *C*.

Q5 When you drag x up $R_{C,\theta}(x)$ moves diagonally up and to the left, and when you drag x left $R_{C,\theta}(x)$ moves diagonally down and to the left.

Q6 The two variables come together when you drag x to the center point. There's no other place where this happens, so the center point is the only fixed point for this function.

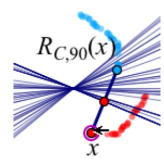
Q7 The new shapes are again the same size, but this time the traced shape made by the dependent variable is only turned half as much (45° counter-clockwise) from the original polygon, compared with the traced shape when $\theta = 90°$.

Q8 When you drag x up $R_{C,\theta}(x)$ moves down, and when you drag x right $R_{C,\theta}(x)$ moves left. The dependent variable always moves the opposite way from the independent variable. The center point is still the only fixed point.

Q9 The independent variable is restricted to the border of the polygon.

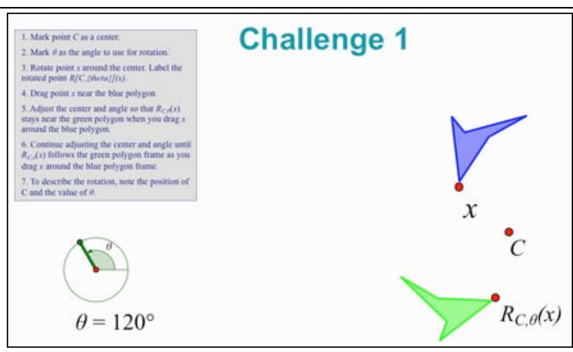
Q10 The range has exactly the same size and shape as the restricted domain, but is rotated from it.

Q11 Answers will vary. Many students will drag to locate the center (the fixed point) and then find the angle by guess-and-check. (A very few may construct the perpendicular bisector of the segment between the independent and dependent variables. For rotation this bisector always goes through the center point, so tracing the bisector while dragging the independent variable reveals the location of the center point. Few students are likely to figure out this technique at this stage of their understanding.)

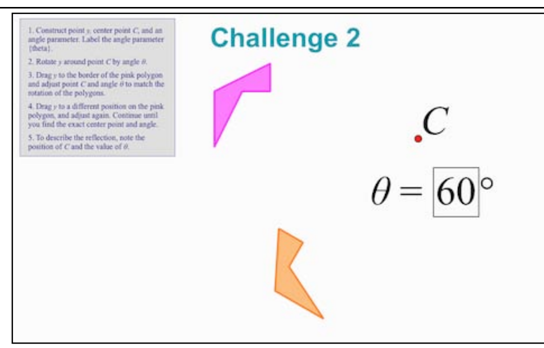


Rotation Challenges

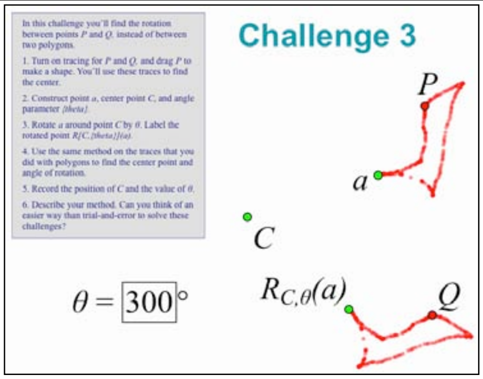
The images below show the location of the center point and the angle by which the figure was rotated for each challenge.



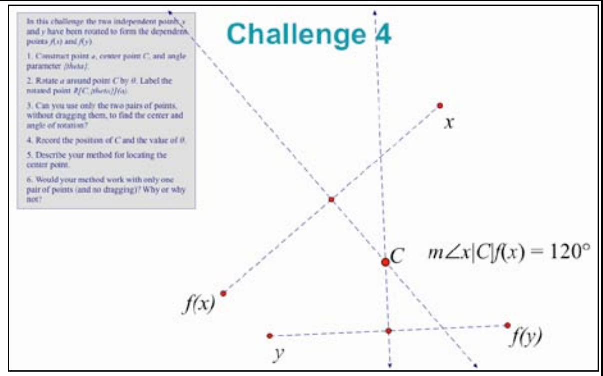
Students will likely approach this problem using guess-and-check methods.



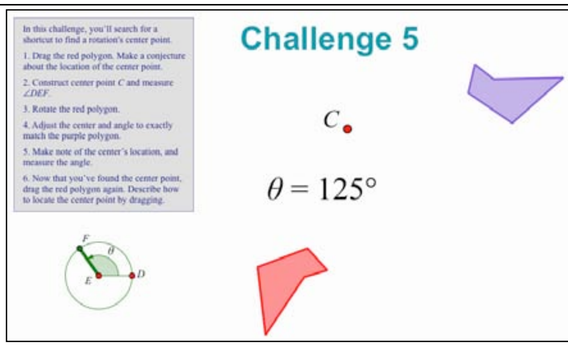
Students may begin to search for a more efficient method than guess-and-check.



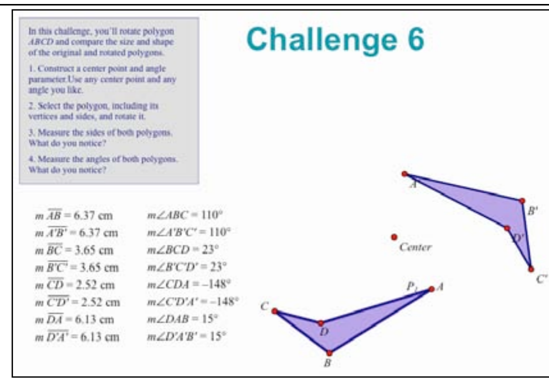
This student has used traces, because a single pair of values does not provide enough information to find the center and angle. It's also possible to find the fixed point by dragging.



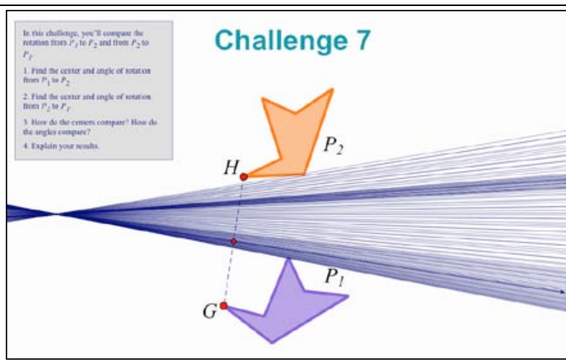
The center of rotation must be equally distant from the corresponding points of the independent and dependent variables, so it must lie on the perpendicular bisector of the connecting segment.



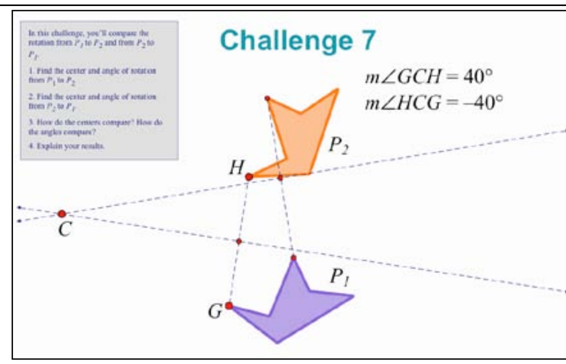
Dragging the two polygons so that corresponding vertices coincide is one approach to finding the fixed point.



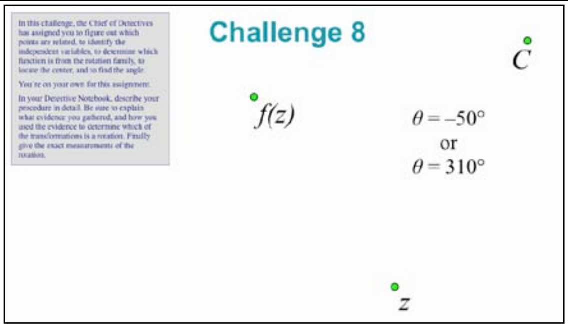
Corresponding sides have equal lengths and corresponding angles have equal measures. (In other words, the polygons are congruent.)



This student constructed the perpendicular bisctor of a segment connecting corresponding points of the two polygons. By tracing the perpendicular bisector while animating the points around the polygons, she revealed the center of rotation.



The concept of inverse functions, suggested by this challenge, will be addressed in a later activity.



Function *f* is the only rotation among the four functions.

Related Activities:

- *ID the Suspects—Identify Functions* (Recommended Prerequisite)
- *Family Resemblances—Identify Function Families* (Recommended Prerequisite)
- *Reflection Challenges—The Reflection Family* (Required Prerequisite)
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Dilation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*

Portions of this material are based upon work supported by the National Science Foundation under award number DRL-0918733. Any opinions, findings, and conclusions or recommendations expressed in this work are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.