Notes for the Teacher

Students explore reflections as functions. They construct a reflection function, label the variables using function notation, and trace and compare the paths of the variables. They restrict the domain to reveal the corresponding range, and they move the mirror to create different members of the reflection family. Finally, students solve reflection challenges by figuring out where the mirror must go to match an existing reflection.

This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Construct an independent variable point *x*, a mirror (which defines the function), and the reflected image of *x* (the dependent variable).
- Label the dependent variable using function notation as meaningful shorthand for the action that created it.
- Drag the independent variable while tracing both variables, and describe both the variables' relative motion (their *covariation*) and the relationship between their traces (their *correspondence*).
- Move the mirror to form and investigate a different member of the same family.
- Restrict the domain of the independent variable to the border of a polygon, and observe and describe the resulting range by dragging the independent variable.
- Identify fixed points of the function, and explicitly compare the relative motion (both speed and direction) of the independent and dependent variables.
- Solve challenges that involve finding a hidden mirror, given an object and its reflection.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites: Though students can use this activity as their first experience with Geometric Functions, it's very helpful for them to have already completed the first two Geometric Functions activities:

- *ID the Suspects—Identify Functions*
- *Family Resemblances—Identify Function Families*

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together in developing students' conceptual understanding of functions.

High Cognitive Demand: This activity provides several tasks for which there is no cut-and-dried procedure for students to follow. Though the worksheet provides fairly explicit directions to help students perform the initial construction, the questions it asks require experimentation, inquiry, and analysis.

Mathematical Habits of Mind, Reasoning and Sense Making: Students construct and investigate mathematical objects, and are challenged to answer questions that require tinkering and analysis to understand the behavior of these objects.

Inquiry: The body of the activity supports student inquiry, and the worksheet contains probing questions that require students to manipulate, observe, and analyze.

Cooperative Learning: Students work in pairs during the exploration portion of the activity, and exchange roles between driving (using the mouse and keyboard) and coaching/recording. Expect students to work purposefully in pairs, to coach each other, and to discuss every part of the activity with their partners. The members of each pair should share the construction and manipulation steps equally, so that each student is fully involved in both creating the function and working with it.

Assessment: You should engage in formative assessment by visiting and questioning student pairs during the exploration phase. You can use the summary discussion to elicit students' understandings, confusions, and questions. The last page of the worksheet is an exit ticket.

Differentiation: The worksheet comes in a long form (with more explicit instructions) and a short form (for students with more experience with Sketchpad). It also includes an optional Answer sheet on which students can write their answers. The concluding Reflection Challenges entail varying levels of difficulty, and are very useful for furthering students' understanding of the Reflection Family and of function concepts in general.

After the class completes this activity, they can progress to similar individual activities for translation, rotation, and dilation functions. Alternatively, they can progress to the Compare Function Families activity in which they create, manipulate, and observe all three of these other families in a more self-directed, project-based way.

Questioning and Discourse: Most discourse will take place between team members during inquiry, so it's important to encourage team members to describe their observations to each other and to discuss their answers to the questions. The questions on the worksheet, along with teacher observations of the work of different teams, should guide the summary discussion. Since the worksheet includes explicit directions for completing the construction, it's important to ask questions that focus students' thinking on the big ideas, both while they are working in pairs and during the whole-class discussion.

Instructional Strategies: By varying *x*, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

The time students will require for this activity can vary significantly depending on their mathematical background and Sketchpad experience. Because the concluding class discussion is critical, you should be prepared to postpone the challenges contained in **Reflection Challenges.gsp** to another day. (You may even elect to make worksheet questions Q8 through Q10 optional.) While you can assign the Reflection Challenges as homework exercises, it's preferable to have students work cooperatively in class while you observe their investigations.

The three stages of the activity are described in more detail below, in the sections on Launch, Inquiry, and Summary.

Worksheet: The worksheet comes in two forms, both of which contain the same questions. The longer form is useful for students who are relatively new to Sketchpad, as it provides explicit steps to follow. If students have experience with Sketchpad, the shorter form is preferable, as it expects more initiative from students and it concentrates more on the mathematics to be explored and less on technicalities. (Giving students unnecessarily explicit directions risks miring students in details, discouraging their thoughtfulness and creativity.) Some teachers may choose to provide one short form and

one long form to each pair of students, telling them to work from the short form, referring to the details in the long form as needed.

Answer Sheet: Many teachers prefer their students to write their answers on their own paper. Alternatively, you might ask students to submit their answers electronically, or you can use the provided answer sheet. If you assign electronic submission, tell students that they can include screen captures by choosing **Edit | Select All** and then **Edit | Copy** in Sketchpad. They can paste the resulting image into a word processor, email, or other document. (The image includes traces and is cropped to the window border, so that students can easily control the area being copied and pasted.)

Transfer of Learning: To get the most value from this activity, students must connect what they've learned to other representations of functions. By dragging the independent variable *x* in this activity, they can relate its smooth motion to the continuous variation of a numeric variable. Moreover, comparing the relative speed and direction of the independent and dependent variables can help students think about the relative rate of change of the variables of a linear function. If students are already familiar with functions in other contexts, you should mention the importance of making connections during the introduction of the lesson, and devote some portion of the summary discussion to these connections.

Launch

Expect to spend about 10 minutes.

To activate students' prior knowledge, start Sketchpad and display page 1 of **Family Resemblances.gsp.** (Students will remember this page from the previous activity. Three of its four functions belong to the reflection family.) Ask a student to drag independent variable *a* while classmates observe the behavior of $f(a)$. Ask students for a short summary of how to recognize when functions come from the same family. (Two or three brief student answers suffice; detailed discussion or review is not helpful at this stage.)

Tell students that today they will explore the reflection function family by creating and manipulating similar functions. They will then investigate the similarities and differences, and use the functions they create to make interesting shapes.

[Optional] If students are completely new to Sketchpad construction techniques, have one or more student volunteers demonstrate steps 1–6 on the long form of the worksheet. Stop the demonstration as soon as the dependent variable has been constructed. (It's most effective to have several students demonstrate one step each. Passing a wireless mouse from student to student can make this easy and effective.) Don't address the questions on the worksheet. Just make sure students know how to create a point, a vertical line, and a reflected image, as well as how to label a point. They can use the directions to find how to change the color of a point and turn on tracing.

[Optional] If students are not yet familiar with the Help system, have another student choose **Help | Using Sketchpad | Sketchpad Tips | Tools | Using the Straightedge Tool.** Tell her to click on the page icon to view the tip in comic form. Tell students that they can always use the Sketchpad Tips or the Reference Center to figure out how to use the program. (Discourage them from using the video icon unless they have headphones attached to their computers.)

Explore

Expect students at computers to spend about 25 minutes.

Assign student pairs to computers and distribute the worksheet. Tell students that the less-experienced partner of each pair should operate the mouse first, and that the other partner should provide coaching and record observations without touching the mouse. Tell students to switch between their roles as operator and coach/recorder after they answer Q5 on the worksheet.

Tell pairs to agree on their answer to each question, and to record their answers in the sketch or on separate paper as you determine.

Circulate as students work, asking them what they notice as they drag the independent variable and observe the dependent variable. Encourage them to drag in various directions and at various speeds as they look for patterns in the relative rate of change of the variables. Make note of questions and difficulties that arise, and encourage students to raise these questions or difficulties during the concluding whole-class discussion.

Whenever possible ask questions instead of providing information. Pay particular attention to students' observations for Q1 ("What happens when you drag point *x* to the right?" or "Is there any direction you can drag to make the variables move in the same direction?") and to their explanations for Q5 ("What would be different if the mirror was at a different angle?" or "Why did you decide to make the mirror horizontal?"). As you check on the groups, plan the order in which to call on students to bring out aspects of function reasoning in a logical way. This can ensure that simple observations precede more sophisticated explanations and interpretations. Also, have selected pairs recall those questions or difficulties that would be beneficial for the entire class to discuss.

If time and technology permit, use a network folder or on a flash drive to collect several student sketches that illustrate interesting difficulties or interesting observations. As you collect sketches, consider the most logical order in which to show them. That way, you can lead a discussion about the relative rates of change of the variables and the shapes traced out when the domain is restricted.

Students who finish early should be encouraged to try **Reflection Challenges.gsp.**

Discuss and Summarize

Expect to spend about 10 minutes.

Gather the class. Students should have their worksheets with them. Give students the opportunity to discuss difficulties or misconceptions.

Summarize the important concepts covered:

- varying the independent variable to observe the behavior of the dependent variable;
- labeling the dependent variables with notation that describes their relationship;
- attending to the relative motion of the two variables and to their fixed points;
- restricting the domain of the independent variable;
- tracing out the corresponding range; and
- comparing the shape of the restricted domain to the shape of the range.

Link the discussion of the shapes to the concept of congruence; this is an opportunity for students to describe in their own words what it means for two figures to have exactly the same shape and size, and to relate the term *congruence* to their own observations.

Call on students to discuss their answers for Q5 and Q8 in the order you determined while circulating. Use these particular students' contributions to try to generate a lively and probing discussion.

In answering Q5 and Q8, students should observe that when they drag the independent variable parallel to the mirror, both variables move at the same speed and in the same direction. However, when they drag the independent variable perpendicular to the mirror the variables move at the same speed but in different directions. To make a connection to linear functions expressed in the form $y = mx + b$, ask students to think of a linear function for which x and y move at the same speed and in the same direction (both increasing or both decreasing), and to think of a different one for which the variables move at the same speed but in different directions (one increasing while the other decreases). (To forge an even stronger and more memorable connection, follow the Geometric Functions activities with Dynagraph activities, which represent independent and dependent variables as points on two parallel number lines.)

Display particular sketches that you collected to illustrate important elements for the discussion. Be sure to validate even incorrect efforts, by emphasizing the value of making mistakes, contemplating them, and persevering in order to learn from them.

In preparation for the Reflection Challenges sketch, ask students to explain how you could find the mirror if it were hidden.

Assess

Just before students leave, ask them to fill out an exit ticket describing one important thing they learned and one thing that they're not sure about.

Review and Challenges

Expect to spend about 30 minutes.

At the beginning of the next class session, project the sketch **Reflection Challenges Review.gsp**. Pages 1 through 4 list the various objectives of this activity, to facilitate a brief review that will help students' retention. On each page have a different student volunteer press the bullet(s) to reveal activity objectives, and then use the page to illustrate that page's objectives.

Page 1: Ask the volunteer to press the bullet and then to press each phrase in italics to reveal in turn the independent variable, the mirror, the dependent variable, and the function notation. Then have the volunteer drag point *x* as students describe how the relative motion varies depending on the direction in which *x* is dragged.

Page 2: After showing the first bullet, have the volunteer drag *x* first vertically (to show the variables moving the in same direction at the same speed) and then horizontally (to show the variables moving in opposite directions). After the second bullet, ask the volunteer to press the italicized phrase *Move the mirror* to put the mirror in a diagonal orientation, thus creating a different member of the same function family. Have students predict the result of moving *x* up and have the volunteer check the prediction. Similarly ask for and test a prediction about the result of moving *x* to the left.

Page 3: Show the first bullet and have the student drag *x* to identify several fixed points. Show the second bullet and have the student merge the independent variable *x* to the polygon. Then have the student drag *x* to show the effect of a restricted domain.

Page 4: This page introduces the reflection challenges from **Reflection Challenges.gsp**. Explain that the first challenge involves finding a hidden mirror that connects the blue polygon to the green one. Having a volunteer do these steps:

1. Drag the independent variable *x*.
2. Restrict *x* to the green polygon.
3. Erase traces and animate *x*.
4. Drag *A* or *B* to adjust the mirror.
5. Erase traces and animate again.

Before the volunteer locates the mirror correctly, stop the demonstration and tell students that discussing and solving these puzzles will develop their intuitive understanding of the reflection family. Send students to the computers in pairs, have them open **Reflection Challenges.gsp,** and ask them to solve as many of the puzzles as they can.

One outcome of working on the puzzles is the ability to look at two reflected shapes, or even at two reflected point variables, and immediately visualize the approximate location of the mirror. Working on the puzzles will help develop this ability, as will connecting the puzzles to other tools for investigating reflections (such as Patty Paper® Geometry or Mira™ Geometry Tools). Once students have this understanding, they can use their visualization of the mirror to develop specific techniques for accurately locating it. They may realize that the midpoint of a segment connecting x and $r_j(x)$ is a fixed point and must lie on the mirror. They may also figure out that the mirror itself must be the perpendicular through that fixed point (in other words, the perpendicular bisector of the segment).

Page 5 of the Challenge sketch has two points, but does not identify which is independent and which is dependent. The challenge asks students to find the mirror that reflects point M onto point N, and then to find the mirror that reflects point N onto point M. Thus the challenge is to find a function that will take the dependent variable back to the original independent variable—in other words, to find the inverse function. (Don't introduce the terms *inverse* or *inverse function* yet. Each transformation challenge activity has a similar question; after answering these questions for several different function families, students will be ready to formalize and name the concept.)

Follow-up

After completing the Reflection Challenges activity, you can take students in either of two directions to explore the rotation, dilation, and translation families.

Individual activities for each family: Students can do the individual function family challenge activities, each of which is similar in structure to this reflection activity. Because students are already familiar with the format, two class periods will be sufficient for students to complete all three activities.

One activity to explore all three families: In the *Family Relationships* activity students explore all three function families in a more independent way, doing their own construction and planning their own investigatory strategy. This puts more responsibility on students for doing the constructions and for planning their strategy.

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes have the unfortunate result of emphasizing and perpetuating them, so it's best if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 Students should observe that when they drag *x* vertically both variables move in the same direction, but when they drag *x* horizontally the variables move in opposite directions.

Q2 Responses will vary. Students may observe (in a variety of terms) that the shape formed by the dependent variable is similar to the shape formed by the independent variable but is flipped horizontally.

Q3 Responses will vary. When the independent variable crosses the mirror, so does the dependent variable, in the opposite direction. The result is that the two variables are always on opposite sides of the mirror.

Q4 The independent variable is described as an "independent object," and the dependent variable is described as the "reflection of Point *x* across mirror *j*."

Q5 Responses will vary. Students will need to try different angles and locations of the mirror to figure out that the mirror must be horizontal. As students work, it may be helpful to point out to them (for instance) that the independent and dependent variables in the picture are vertically aligned, and to ask them how they can adjust the mirror to align their variables vertically.

Q6 Responses and drawings will vary, but in every case students should indicate that the fixed points lie on the mirror.

Q7 Responses will vary depending on the orientation and location of the mirror.

Q8 The function's domain is the boundary of the polygon.

Q9 The points move at the same speed. They move in the same direction when they move in a direction parallel to the mirror, in opposite directions when they move toward or away from the mirror, and at similar "flipped" angles when they move in other directions.

Q10 The range is the same shape as the domain, but flipped. Students should explain how the different directions in which the variables move results in flipping the picture.

Q11 Students' explanations will vary. With a known reflection, students can identify corresponding features (for example, corresponding vertices). A fixed point is located exactly halfway between such a pair corresponding features, so if students can identify two pairs of corresponding features, and then two fixed points, they can locate the mirror.

Related Activities:

- *ID the Suspects—Identify Functions (Prerequisite)*

- *Family Resemblances—Identify Function Families (Prerequisite)*
- *Rotation Challenges—The Rotation Family*
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Dilation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*
- *Transform Twice—Function Composition*
- *Compose a Locus—Composition, Domain, and Range*
- *Special Effects—A Swirling Transformation*
- *Animated Special Effects—Swirl a Picture*

License (CC-BY-NC-SA 3.0)

Portions of this material are based upon work supported by the National Science Foundation under award number DRL-0918733. Any opinions, findings, and conclusions or recommendations expressed in this work are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.