Notes for the Teacher

This activity introduces relations and functions by treating geometric points as the independent and dependent variables. Students explore and describe relationships between point variables, distinguish between independent and dependent variables, and use examples and non-examples to formulate a definition of *function*.

Ideally these activities are students' formal introduction to *function* and its related concepts; the dynamic sensory-motor nature of these experiences, and the vivid, concrete way that geometric transformations embody functions, provide solid grounding on which to develop robust conceptual understanding. Even if students are already familiar with functions in the numeric realm, these activities create a concrete sensory-motor underpinning, allowing them to consolidate concepts that were introduced in a more symbolic and abstract form.

This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Classify points as independent or dependent.
- Drag independent points and observe and describe the resulting behavior of dependent point(s).
- Use the drag test to distinguish functions from non-functions.
- Formulate and discuss their own definition of *function*.
- Use the term *relation* to describe two variables that have a relationship that may or may not be a function.
- Construct their own examples of functions and non-functions (optional).

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

critique the reasoning of others; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Background:

Several of these objectives are particularly important for the activities that follow, and for developing a deep understanding of function. In particular, cognitive science researchers observe that students absorb abstract concepts through sensory-motor experiences such as those incorporated into these activities.

Drag independent points: Explorations of geometric functions make consistent use of points as the independent and dependent variables. An important advantage of this approach is the ease with which students drag the independent variable to observe the behavior of the function. As they drag, students exercise full control of the speed and direction of the independent variable, and can determine whether each location (value) of the independent variable corresponds to a single location (value) of the dependent variable.

Observe behavior: As students control the speed and direction of the independent variable, they observe the speed and direction of the dependent variable. Because they are in control of the speed and direction, students can make their observations as quickly or slowly as they like, and can refine their observations by moving more slowly or by repeating a motion.

Describe behavior: Students are encouraged to notice and describe whether both variables move in the same or different directions, and whether they move at the same or different speeds. Such observations of relative rate of change are critical to understanding function behavior, to differentiating one function family from another, and to eventually to understanding calculus.

Use the drag test: The *drag test* is a general term for using dragging (variation) as a tool for revealing and analyzing important mathematical behavior. Its importance derives from the way it engages students' motor faculties as they drag an object and engages their visual sense as they observe and analyze the changes in one or more other objects.

Defining and identifying functions: Student discourse, both with their partners and in the full class discussion, is critical for building a solid understanding of functions. During the group discussion it's advantageous, when possible, to refer back to specific pages of the sketch and to the specific behavior of functions and non-functions. This will directly relate the concept to students' dragging actions and their observations of the consequences.

On your own: Students will enjoy building their own functions and non-functions for their classmates to categorize. This act of construction consolidates their understanding of the function concept. Building functions will be easier for students who already have experience with creating Sketchpad constructions. For students without past construction experience, page 12 gives explicit step-by-step instructions that most students will be able to follow. Later activities will focus on such constructions.

Vocabulary:

Independent point and *dependent point:* The activity uses these terms interchangeably with *independent variable* and *dependent variable*. Using a common vocabulary, in which variables may be numbers or may be points, makes it easier for students to connect the functions they work with in a geometric context with those they work with in an algebraic context.

Relation: The activity uses the term *relation* to denote the relationship between points. There's no need right now to provide a precise definition of *relation*. It's enough for students to understand that variables may be *related* to each other, and that *relation* has a more precise mathematical definition that they'll learn later.

Function: One objective of this activity is to agree on a definition for function. This should be the focus of the closing class discussion. During that discussion you should avoid imposing a particular definition, but instead strive for clarity and consensus on a definition generated by students. The class will refine this preliminary definition as they engage in later activities.

Introduce:

Use a projector to show sketch page 1 of **ID the Suspects.gsp.** Distribute the worksheets and ask a student to read the directions. Call another student to the interactive whiteboard (if you have one) or to the computer to drag point a. Ask students to describe the result; they will observe that a and y are related. Ask students what they think will happen if y is dragged, and then ask a student to actually drag y to see what happens.

Tell students that they will record their observations on the grid labeled Q1 on their worksheet, and ask several students to explain in their own words which variable should be listed as independent and which as dependent. Students can then fill in the top left cell of Q1 to show $a \rightarrow y$.

If students are already familiar with independent and dependent variables in the context of linear functions, take a moment for discussion of the relationship between numeric and geometric variables. Students may observe that they can substitute a number for the value of independent variable a in an equation like $y = 2a + 3$ and calculate a result for the dependent variable y. By asking students how they can vary a in the linear function, and how they can vary a in the geometric function, encourage them to realize that they can vary the independent variable a in either type of function and see what effect the change has on the dependent variable y.

Have the student at the computer continue dragging, and ask her to try to drag a and y to the same location. Ask for suggested descriptions of the relationship. Students may say:

- *When I drag* a *to the right,* y *moves left and slightly up.*
- *When I keep dragging* a *to the right,* a *and* y *come together in the middle of the screen.*
- *When I drag* a *up,* y *moves up and slightly to the right.*

- *There are many places where I can drag* a *to put it together with* y. *These places seem to lie along a straight line.*
- *Variable* y *moves at the same speed as* a, *but in a different direction.*

Make sure that students state observations that address the two variables' relative speeds and their relative directions. Lead them to seek locations where the two variables come together. Introduce the term *fixed point* for such locations. Ask a student to drag point *a* to locate as many fixed points as possible. Use your questioning to encourage students to notice and describe the relative speed and direction of the variables and the existence, locations, and patterns of fixed point(s) throughout this activity.

Have students fill in the top right cell of the Q1 grid on their worksheets with one of the observations made during the class discussion.

Ask another student try to drag *b*. When *b* doesn't move, ask students what they think this means. Don't evaluate these conjectures; students will have time later to evaluate and refine their conjectures. Have the same student drag *c*, and ask students how they should fill in the second grid cell on the left ($c \rightarrow d$).

Tell students that they will have time later to drag the variables themselves to complete the rest of the table on their worksheets. Go to page 2 and have another student read the directions. Tell the class that their job is to develop their own definitions of a function. Point out that the directions say that *x* and *x'* are an example of a function, and have a different student drag *x* while the class observes the result. Then have the student drag *y* to observe the behavior of a non-example. Tell the class that they'll have a number of examples and non-examples, and that their job is to work with their partner to describe the behaviors of the various functions and non-functions and to develop and write the clearest definition of a function that their team can formulate.

Explore:

Assign students to partners, send them in pairs to the computers, and have them open **ID the Suspects.gsp.** Ask them to finish identifying independent and dependent variables on page 1 and then work through pages 2 through 11. On each page they should describe the behavior of the function and the non-function. As they work through the pages, students should use the differences they observe to develop a definition of a function. As you circulate, observe how students interact with and write about the features of functions and non-functions that appear on various pages. (Some of the particularly important features are mentioned in the Discuss and Summarize notes below.) Ask students probing questions about their observations and descriptions, and make mental notes of items you want to address and students you want to call on during the class discussion.

Tell students that pages 11 and 12 are optional, but are a nice challenge if they have finished with the earlier pages and have shared their written definition of a function with you.

As you circulate, make sure students are writing clear descriptions of the behavior they observe using complete sentences. Students should observe when only some locations of the independent variable produce multiple dependent variables. If students initially think that both relations on a given page are functions, tell them to drag the independent variables to more locations. As students finish their definitions, assign several to type their definitions on your projection computer (with the projector turned off) so that you can easily display them later.

Discuss and Summarize:

When students have had enough time to investigate and describe the functions and non-functions on pages 2 through 11, call them together for a class discussion. (Always have monitors turned off or laptop lids closed for such discussion.)

Begin by asking students which pages they found most interesting. Below are some observations that students might make about the various pages. Not all of these observations may come up in the discussion of this activity. However, some of the observations not specifically discussed today may prove useful as you make connections between Geometric Functions and other types of functions in later activities.

- *On page 1, one of the independent variables has no dependent variable, and one has two dependent variables.*
- *Several pages have arrows connecting independent and dependent variables.* Ask students what they think the arrows mean. (The arrows are a graphic embellishment intended to make the functional dependence more concrete; it's important that students not attribute inappropriate mathematical significance to them.)
- *On page 4, the dependent variable seems to fragment into 5 different variables when you drag in certain directions.*
- *On page 5, point b' never moves.* Students may want to say that it's not a relation. This can generate a nice discussion about the definition of a function. One possible question is this: "Try writing your definition of function both ways, including and excluding this case. Which definition is simpler?"
- *On pages 6 and 7, the independent variables have to be dragged quite far before the dependent variable splits into two.* Students may wonder how this is related to whether they are or are not functions. A good response might be that there are times when we can treat a relation like a function when that's the behavior we observe. In a real-life example, we may observe that all shadows formed by the sun at a particular time of day fall in the same direction, but that a shadow formed by a street light varies in direction according to the location of the object casting the shadow.
- *On page 8, the dependent variables seem to jump around rather than move in curved or straight paths.* That is, they move discretely rather than continuously. You may ask students what is different about the behavior of dependent variables on this page than on other pages. Encourage a variety of descriptions of the discontinuous behavior. You might ask students to compare this function to the constant-function behavior of *b* and *b*' on page 5. (The function on page 8 behaves like a constant function when you drag the

independent variable a short distance — but as you continue the dependent variable suddenly jumps.

- *On page 9, the arrows were absent, but the behavior of the dependent variables could still be observed.* Ask students whether the arrows made any difference. Did they press the *Hide Arrows* button? Did the points behave differently afterward?

- *On page 10, dragging t has no effect on dependent variable t'.* (This is similar to page 5, but this time without an arrow to provide a hint.) Students may question whether this is a function at all, since they see no relationship. This can lead to a nice discussion of whether one should exclude one particular kind of behavior (not moving) from the behaviors of interest. Dealing with this question now will help students understand that constant functions are indeed functions.

- *On page 11, the dependent variables appear to move in circles.* .Indeed, both trace out the shape of a circle, one by tracing the entire circle, and the other by tracing out each branch separately.
(If students are already familiar with the vertical line test, they may bring the question up now, because the trace of *n'* would not pass the vertical line test. This is an opportunity to stress that the vertical line test is specific to a particular representation, numeric values plotted on a Cartesian graph, and to emphasize that the real test is not whether a vertical line intersects a Cartesian graph, but whether two or more different values of the dependent variable exist for a single value of the independent variable.)

Ask students to identify common threads they observed as they explored functions and non-functions. Possible observations:

- *The behavior of non-functions was more interesting than the behavior of functions.*

- *Functions always have a single output variable; never more than one and never less than one.*

- *What if a relation sometimes has one output value like a function, and sometimes has no output value at all?* The definition of a function declares that there's exactly one output value, so relations are hard cases. (They may even prompt mathematicians to take extreme measures like restricting the domain or inventing new kinds of numbers.)

Display definitions from various teams. Encourage students to debate the merits of the various definitions. Ask questions of your own to clarify students' thinking and to home in on the critical aspects of their definitions. Discuss and list criteria for a good definition. Formulate a class definition that all can agree on, drawing on as many ideas as possible from the proposed definitions. Have students copy the class definition(s) to their journals. Students haven't been asked to define relation at this time, nor have they seen non-examples.

A useful concluding question, if time permits: Why do you think mathematicians consider it so important to study functions?

Review:

Use the first part of the next class session to review the experience for students. Open **ID the Suspects Review.gsp** and go to page 1. On each page, click the bullets in turn, and ask students to comment on and discuss the summaries that appear.

Definition: This page reviews the work that students did in formulating their definitions.

Notation: This page introduces a new way of writing a function. Instead of representing a function by an arrow between the variables ($a \to y$), you can give a name (such as f) to the operation that the function performs. Then a stands for the independent variable, f stands for the operation, and $f(a)$ stands for the result of performing operation f on variable a. This notation is introduced here because it's used on the next two pages.

Behavior: This page reviews the behavioral characteristics that students observed and investigated as they distinguished between functions and non-functions. The examples all make use of function notation, and care in observing their behaviors will be important in the next activity.

Preview: This page previews the next activity (*Family Resemblances*) by presenting four functions, three from the same family and one from a different family.

Answers:

Q1 Student descriptions of the relationships will vary; different students will record different aspects of the function's behavior. The grid below identifies the actual transformations used, which can be helpful when evaluating students' recorded answers.

Independent Point	Dependent Point(s)	Relationship
$a \to y$		Point y is a reflection of a across a hidden line.
$z \to b$		Point b is a 180° rotation of z about a hidden point.
$c \to d$		Point d is a translation of c.
$v \to$		Point v has no relationship to any other point.
$w \to e, x$		Point e is a dilation of w with respect to a point. Point x is a translation of w.

Q2 There is only one x', but there are two y's.

Q3 There is only one b', but there are two a's.

Q4 On pages 4 through 11, students list the function and the non-function. Students' observations and questions will vary; some possible responses are shown.

Page	Function	Non-function	Observations and Questions
4	$q \to q'$	$p \to p'$	There are multiple p's.

5	$b \to b'$	$a \to a'$	There are multiple *a*'s. Even though $b \to b'$ is a functions, *b'* never moves!
6	$c \to c'$	$d \to d'$	Depending on how *d* is dragged, there may be either one or two *d*'s.
7	$v \to v'$	$u \to u'$	Depending on how *u* is dragged, there may be either one or three *u*'s.
8	$s \to s'$	$r \to r'$	Both *r*' and *s*' move in a jerky way. Depending on how *r* is dragged, there may be either one or two *r*'s.
9	$w \to w'$	$z \to z'$	Depending on how *z* is dragged, there may be either one or two *z*'s.
10	$t \to t'$	$e \to e'$	There are two *e*'s.
11	$n \to n'$	$m \to m'$	Both *m*' and *n*' move in circles or parts of circles, but there are two *m*'s, and only one *n*'.

Q5 In a function, for every location of the independent variable there is exactly one location for the dependent variable.

List of Transformations

This list summarizes the transformations on each page of **ID the Suspects.gsp.**

Page	Function	Behavior
1	$a \to y$	reflection
1	$c \to d$	translation
1	$w \to e, x$	dilation by scale factor of 2, translation
1	v	none
1	$z \to b$	rotation by 180° or dilation by a scale factor of –1
2	$x \to x'$	reflection across horizontal mirror
2	$y \to y'$	reflection & rotation
3	$a \to a'$	translation & dilation
3	$b \to b'$	translation
4	$p \to p'$	a reflection & four rotations
4	$q \to q'$	rotation by 180° or dilation by a scale factor of –1
5	$a \to a'$	three different transformations
5	$b \to b'$	constant transformation

6	$c \to c'$	glide reflection
6	$d \to d'$	glide reflection & translation
7	$u \to u'$	three transformations, two of which appear only when u is on the right half of the screen
7	$v \to v'$	translation
8	$r \to r'$	two different discrete transformations
8	$s \to s'$	discrete transformation
9	$w \to w'$	one transformation
9	$z \to z'$	two transformations, one of which appears only when z is near the bottom of the screen
10	$e \to e'$	two dilations
10	$t \to t'$	constant transformation
11	$m \to m'$	two transformations that together trace out a circle
11	$n \to n'$	one transformation that traces out a circle

Related Activities:

- *Family Resemblances—Identify Function Families*
- *Reflection Challenges—The Reflection Family*
- *Rotation Challenges—The Rotation Family*
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Dilation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*

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