# Notes for the Teacher

This activity introduces the idea of families of functions, treating geometric points as the independent and dependent variables. On each page students explore four functions to identify which one is from a different family than the other three.

By exploring the behavior of these functions, students become familiar with functions that use translation, rotation, dilation, reflection, and glide reflection, and they develop an ability to distinguish the behavior of one family of functions from another. This activity prepares students to create their own functions in subsequent activities. By classifying functions based on their behavior – particularly their relative rates of change and their presence or absence of fixed points – students conceptualize function families based on their characteristic relationships between their inputs and outputs. Students can extend this understanding to algebraically-represented functions, characterizing them in terms of the relative rate of change of their variables and of their special points (such as fixed points, intercepts, and critical points).

Students are encouraged to use a variety of criteria to compare the behaviors of functions:

- the relationship between the shapes traced by the two variables;
- the relative directions in which the variables move;
- the relative speeds of the variables; and
- the presence, location, and pattern of fixed points.

This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

## *Objectives:*

In this activity students will:

- Drag independent variables and observe and describe the behavior of dependent variables. (Use the drag test.)
- Distinguish between functions based on the shapes traced by the variables.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

  

- Distinguish between functions based on the relative speed and direction of the variables.
- Distinguish between functions based on the presence and pattern of fixed points.
- Describe several function families (such as translations, rotations, dilations, reflections, and glide reflections) based on the similarities and differences of their behaviors.
- Formulate a working definition of a function family.
- Determine whether a function belongs to a particular family.
- Use function notation appropriately to refer to a function and to a dependent variable.
- Construct new functions belonging to particular families (optional).

Many of these objectives will appear again when students learn about families of numeric functions. In particular, they'll find it useful to attend to relative speed and direction (related rates) and to special points of numeric functions (fixed points, intercepts, critical points).

***Common Core Mathematical Practices:*** (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

***Common Core State Content Standards:*** 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

***Grade Range:*** Grades 7–11

## ***Background:***

*Exploration and Discussion:* The heart of this activity is the exploration and discussion in which students engage as they identify and describe function families. By exploring and discussing with classmates, students sharpen their observations of function behavior, identify similarities and differences between two functions, and refine their understanding of what a function is. These outcomes are not only tools to support astute analysis and correct classification of function families, but stand on their own as important goals of the activity.

*Explore More:* Some students will enjoy constructing their own functions that belong to different families. They can then challenge their classmates to investigate their functions and identify the families to which they belong.

The Explore More construction will be easier for students who already have some experience with construction in Sketchpad. For students without such experience, page 12 gives fairly explicit step-by-step instructions that many students will be able to follow. Later activities will focus on such constructions.

 

## ***Vocabulary:***

*Function Family:* On each page of the sketch, students should be able to observe and explain characteristics shared by three of the functions but not by the fourth. Encouraging students to create problems of their own, either during class or—if possible—as homework, will help them to understand how the mathematical construction of a function determines its behavior. Eventually students should come to understand that functions with similar constructions show similar behavioral characteristics, and will be able to connect the function-family concept to both how functions are defined and how they behave. This understanding will develop over the next several activities.

*Function Notation:* In the earlier activity *ID the Suspects—Identify Functions*, students were exposed to function notation such as $b \to b'$, indicating that the independent variable is $b$, the dependent variable is $b'$, and that $b'$ depends upon $b$. In this activity students encounter function notation in which $z$ is the independent variable, $h$ is the name of the function, and $h(z)$ is the dependent variable. Naming a function encourages students to think of it as an object worthy of study in its own right. The objects of study in this activity are not only the variables (which have names), but also the relationship between the variables (the function, which has its own name). This notation is introduced in a way that is both natural and useful for students' work. There's no need to point the notation out ahead of time, but during discussion (both individual and whole-class) you should expect students to use function notation correctly and appropriately.

## ***Introduce and Model:***

Project page 1 of **Family Resemblances.gsp.** Pass out the worksheet and ask a student to read the directions. Call another student to the interactive whiteboard (if you have one) or to the computer to drag each of the points on page 1.

Ask the class to describe what they observe about the behaviors of points, both those that can be dragged and those that cannot. Students should classify the points as independent and dependent variables, and related pairs of points as functions. Encourage students to observe that the color and the notation are hints that make it easy to tell which variables are related. The notation also suggests which of each pair of variables is independent and which is dependent. Tell students to write answers to worksheet questions Q1 and Q2 in their own words.

Draw students' attention to the table in Q3, and tell them that they'll fill in the first row of Q3 as a class before they start working in pairs. Ask them what variable they should drag to investigate function $f$. Review briefly the notation used here: the variables are called $a$ and $f(a)$, and the function is called $f$. Try to elicit (or suggest, if necessary) the idea that $f(a)$ is the output that results when input variable $a$ has been acted on by function $f$. Though it's not uncommon to see a function referred to as $f(a)$, this is poor practice. It's important for students to clearly distinguish the function $f$ and the dependent variable $f(a)$.

Have the student at the computer drag variable $a$, and ask students to describe the behavior of this function. It's important not to use questioning techniques that lead students to particular answers. Instead, encourage students to paraphrase each other and to build on each other's descriptions. As students describe various aspects of the function's behavior, encourage them to pay attention to features that will be important when they describe the behavior of symbolic/numeric functions:

- the *relative directions* in which the variables move,
- the *relative speeds* with which they move, and
- the existence and location of any *fixed points*. (A *fixed point* is a location at which the independent and dependent variables coincide.)

Remind students that they observed these three characteristics (*relative direction* of motion, *relative speed* of motion, and *fixed points*) in the previous activity. Stress how important it is to observe these characteristics as they look for family resemblances in today's activity.

Try to let students themselves suggest trying the *Tracing On* and *Tracing Off* buttons to see whether they are useful. Once students have turned traces on, you may want to ask them to think about how the traces might help to reveal relative direction, relative speed, and fixed points.

Following this discussion, ask students to fill in the first row of the table, using their own words and writing in complete sentences.

## *Explore:*

Assign students to partners, send them in pairs to the computers, and have them open **Family Resemblances.gsp.** Ask them to finish filling in the table for Q3, and then work through pages 2 through 11. For each page they should describe, in complete sentences, the behavior of the three functions that are similar and the behavior of the one function that is different.

Tell students that pages 12 through 14 are optional, but are a nice challenge if they finish the earlier pages.

As you circulate, verify that students are writing clear descriptions, in complete sentences, of the similarities and differences they observe. As you look at their descriptions, note which ones will be useful in naming the different families during the class discussion. Observe students' dragging strategies for figuring out which function is different, and encourage them to drag purposefully.

As you circulate, identify students with particularly interesting or revealing observations and strategies. Later, during the discussion, call on these students to describe what they noticed or what strategies they used. (For instance, a student might drag all four independent variables at the same time, to compare how the dependent variables move as she moves the mouse.)

## *Discuss and Summarize:*

When students have had enough time to investigate and describe the similar and dissimilar behaviors on pages 1 through 11, call them together for a class discussion. (Always have monitors turned off or laptop lids closed for such discussions.)

 

Begin by have individual volunteers come to the computer and demonstrate pages that they found most interesting. Have them demonstrate, and describe in their own words, the similarities that they observed in three of the functions on that page. Then suggest that the class invent a name of the family being demonstrated. Suitably descriptive names might be *slide*, *follow*, or *keep pace* for a translation, *turn*, *twist*, *circle*, or *spiral* for a rotation, *stretch*, *shrink*, or *scale* for a dilation, *flip*, *reflect*, or *mirror* for a reflection, and *flip-and-slide* or *step* for a glide reflection. (Though *spiral* doesn't describe the behavior of a rotation, it does describe the trajectories when students move the independent variable toward the dependent variable.)

Ask students what strategies they used to find the function that was different. How much did it help to use the traces? What did they notice about how some traces differed from others? Is there any connection between the relative shapes of the traces and students' observations about relative speed, relative direction, and fixed points of a function?

If there's time, have the students agree on a class name for at least the translation, rotation, dilation, and reflection families. For each such family, have students identify its behavioral features. (For example, the relative rates are the same for all but dilation. Direction of the motion of both variables is the same for translation and dilation, always different for rotation, and sometimes different for reflection. Rotation and dilation have a single fixed point, reflection has a line as a fixed point, and translations don't have a fixed point at all.)

Pages 9, 10, and 11 use pictures as the input and output to the various functions. Ask students how they found these pages different from the previous pages. Students may say that it was easier to analyze the pictures. Question them as to why they found it easier. What extra information, if any, did the pictures provide?

Then ask if these pages used any of the families that they just named and described. Ask students what other connections they saw between the pages with variables and the pages with pictures. What would they say about the independent and dependent variables on these pages? (There are several ways to answer this question; its main value is to provoke discussion. One answer is to say that the entire first picture is the independent variable, and the entire second picture is the dependent variable. Another is to say that the first picture is made up of many points, each of which is an independent variable, and that all of these points are transformed to make the second picture. The first approach gets into complications; you may want to observe that many questions yet to come will be easier to think about if variables are simple points or numbers, rather than complex pictures or entire sets of numbers.) Students also may observe that the second picture can be thought of as the trace of the dependent variable that would result from dragging an independent variable along a path that forms the first picture. Students may also observe that if you dragged the independent variable along a path to trace out the first picture, the dependent variable would trace out the second picture.

At the beginning of the next class period, you can use **Family Resemblances Review.gsp** to remind students of the big ideas from this activity and to give them a quick preview of the next activity (by demonstrating how to construct and label a member of the reflection family).

 

## *Answers*

The answers below are summaries, or representative answers. Student answers will vary considerably.

**Q1** The independent variable has a single letter label, and the dependent variable has one letter and then another in parentheses.

**Q2** The label of the independent variable appears inside the parentheses for the related dependent variable.

**Q3** (Functions *f*, *h*, and *j* are reflections, and *g* is a dilation. Student answers should include observations such as those below.)

| Function | Behavior |
|---|---|
| ***f*** | The variables move at the same speed, sometimes in the same direction and sometimes in different directions. There are fixed points that seem to be lined up with each other. The traces look like they're the same shape, but flipped. |
| ***g*** | The variables move in the same direction, but the dependent variable moves more slowly. There is one fixed point. The traces are the same shape, but the trace of the dependent variable is smaller. |
| ***h*** | The variables move at the same speed, sometimes in the same direction and sometimes in different directions. There are fixed points that seem to be lined up with each other. The traces look like they're the same shape, but flipped. |
| ***j*** | The variables move at the same speed, sometimes in the same direction and sometimes in different directions. There are fixed points that seem to be lined up with each other. The traces look like they're the same shape, but flipped. |

**Q4** For *f*, *g*, and *h*, the variables always move at the same speed and in the same direction. For *j*, they move in the same direction, but the dependent variable moves faster. (Functions *f*, *g*, and *h* are translations, and function *j* is a dilation.)

**Q5** For *f*, the fixed points lie along a line halfway between the initial locations of the variables. For *g*, *h*, and *j*, there's a single fixed point at the center of the turn or rotation. (Functions *g*, *h*, and *j* are rotations, and function *f* is a reflection.)

**Q6** For *p*, the dependent variable always moves at exactly the same speed and in the same direction as the independent variable. For *q*, *r*, and *s*, when you drag the input toward or away from the output, the variables move in the opposite directions from each other. When you drag the input sideways compared to the output, the variables move in the same direction. (Functions *q*, *r*, and *s* are reflections, and function *p* is a translation.)

**Q7** For *f*, *g*, and *h*, the variables always move at the same speed but in different directions. For *f* and *g*, the directions are only slightly different; for *h*, they are very different — approximately perpendicular. For *j*, the variables move in the same direction, but the dependent variable moves slightly faster. (Functions *f*, *g*, and *h* are rotations, and function *j* is a dilation.)

**Q8** Pages 6, 7, and 8:

| Page | Fn | Describe the difference |
|---|---|---|
| **6** | *g* | The input of *g* can't ever reach the output, so there are no fixed points. The other functions are reflections, and *g*'s variables move like a reflection, but they are offset from each other. (This is a glide reflection.) |
| **7** | *r* | The other functions are rotations. But for function *r*, the output always stays the same distance and direction from the input. (This is a translation.) |
| **8** | *g* | The output is always the same turn counter-clockwise from the input. (This is a rotation.) The other functions have more rotation the farther they get from the moon. |

**Q9** Pages 9, 10, and 11:

| Page | Different Function | Describe the difference |
|---|---|---|
| **9** | *h* | Function *h* is a translation; the others are all reflections. |
| **10** | *gazelle* | The gazelle is slightly smaller and pointed the same direction (dilation). The other animals are all turned (rotations). |
| **11** | *butterfly* | The butterfly is reflected; you can move the input and output together. The others are translated as well as reflected (glide reflections). |

## *Related Activities:*

- *ID the Suspects—Identify Functions* (Prerequisite)
- *Reflection Challenges—The Reflection Family*
- *Rotation Challenges—The Rotation Family*
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Dilation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*

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Portions of this material are based upon work supported by the National Science Foundation under award number DRL-0918733. Any opinions, findings, and conclusions or recommendations expressed in this work are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.