Notes for the Teacher

Description:

Students create, explore, compare, and contrast three different families of geometric functions. By varying the independent variable, observing the functions' behavior, and restricting the function's domain, students identify distinguishing characteristics (such as relative rate of change and locations of fixed points) that are shared by members of a particular family, and that distinguish that family from other function families.

Students construct each family largely on their own, without explicit step-by-step directions to guide them through the constructions, so it's best if they have already completed the Reflection Challenges activity.

> This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Create two members each of three different function families (rotation, dilation, and translation).
- Investigate the behavior of each function by dragging the independent variable, observing the dependent variable, and noting patterns of covariation (by observing relative rates of change of the variables) and correspondence (using traces to observe patterns of corresponding variable locations).
- Identify the location(s) and patterns of fixed points (where the independent and dependent variables coincide) that characterize each family.
- Restrict the domain of an independent variable to a polygon, and observe the resulting range of the dependent variable.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Use function notation appropriately, in written work and during discussions, to identify functions and their dependent variables.
- Identify the behavioral characteristics shared by the members of each function family.
- Use differences in behavior to distinguish each function family from the others.
- Create and describe a new function family (glide reflection) and add it to Sketchpad's Transform menu (optional).

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

- Reflection Challenges—The Reflection Family (Prerequisite)
- ID the Suspects—Identify Functions (Recommended)
- Family Resemblances—Identify Function Families (Recommended)

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together in developing students' conceptual understanding of functions.

High Cognitive Demand: This activity provides tasks, including the initial construction, for which there is no cut-and-dried procedure for students to follow. The questions asked on the worksheet are at a high level that requires experimentation, inquiry, and analysis.

Mathematical Habits of Mind, Reasoning and Sense Making: Students construct and investigate mathematical objects, and are challenged to answer questions that require tinkering and analysis to understand the behavior of these objects.

Inquiry: The body of the activity supports student inquiry, and the worksheet contains probing questions that require students to manipulate, observe, and analyze. There are interesting discoveries for students to make as they work: What happens when you drag the independent variable toward the dependent variable? Do the domain and range exhibit similarity or congruence?

Cooperative Learning: Students work in pairs during the exploration portion of the activity, and exchange roles between driving (using the mouse and keyboard) and coaching/recording. Expect students to work purposefully in pairs, to coach each other, and to discuss every part of the activity with their partners. The members of each pair should share the construction and manipulation steps equally, so that each student is fully involved in both creating the function and working with it.

Assessment: You should engage in formative assessment by visiting and questioning student pairs during the exploration phase, and use the summary discussion to ascertain what students understood and where their confusions are. The last page of the worksheet is an exit ticket.

Differentiation: The open-ended nature of the activity, and the optional Explore More section on creating and analyzing a glide reflection function, are designed to engage students with different levels of background knowledge and to encourage self-directed work.

Questioning and Discourse: Since most discourse will take place between team members during inquiry, it's important to encourage them to describe and explain their observations to each other, and to discuss their answers to the questions. The questions on the worksheet, along with teacher observations while visiting different teams, should guide the summary discussion. It's important to ask questions that focus students' thinking on the big ideas, both while they are working in pairs and during the concluding class discussion.

Instructional Strategies: By varying the independent variable, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

To construct and investigate all three of these function families will take students about two class periods (approximately 90 minutes), assuming that they have already completed the prerequisite Reflection Challenges activity. The time will vary depending on students' past Sketchpad experience. If you expect students to need a fair amount of direction, it's better to do the separate activities on the individual transformation functions, which cover the same ground but with much more explicit directions.

The time students will actually require for this activity also can vary significantly depending on their mathematical background and comfort with self-directed inquiry. Because the activity is designed to require student initiative and creativity, you should pay special attention to which students you assign to work together. You should also remind students to keep the big picture in mind as they investigate and compare three different function families.

Worksheet: From the Worksheet document, duplicate for each student the two worksheet pages. At the same time, print the two Function Family Characteristics pages on which students will summarize their work. The worksheet expects students to be self-reliant and concentrates on the mathematics to be explored rather than on explicit directions for using Sketchpad.

Construction Hints: Print several copies of the Construction Hints page (included in the Worksheet document) to place around the room for students to refer to as they work.

Answer Sheet: Many teachers prefer students to write answers on their own paper. Alternatively, you might ask students to submit their answers electronically, or you can use the provided answer sheet. If you assign electronic submission, tell students that they can include screen captures by choosing **Edit | Select All** and then **Edit | Copy** in Sketchpad. They can then paste the image into a word processor, email, or other document. (The resulting image includes any traces, and is cropped to the window border, so that students can easily control the area being copied and pasted.)

The three stages of the activity are described in more detail below, in the sections on Launch, Inquiry, and Summary.

Transfer of Learning: To get full value from this activity, students must be able to connect what they've learned to other representations of functions. By dragging the independent variable x in this activity, they can relate their dragging actions to the continuous variation of a numeric variable. They can also connect the relative speeds and directions of independent and dependent point variables in a geometric function to the relative rate of change of numeric variables in a linear function. If students are already familiar with the function concept in other contexts, you should mention the importance of making connections during the introduction of the lesson, and devote some portion of the summary discussion to these connections. For instance, in the class discussion of relative rate of change, students should observe that the dependent variable moves at the same speed and in the same direction as the independent variable for any translation. You might follow up with an example of a translation of 2 cm at 90°, elicit the observation that the dependent variable is always 2 units higher than the independent variable, and ask students if they can describe a linear function (expressed as $y = mx + b$) for which the value of y is always 2 units greater than the value of x ($y = x + 2$). Then ask students to give another linear function example that corresponds to a translation of 3 units at 270° ($y = x - 3$).

Through these questions, and similar questions related to dilations, students can connect the constant term b of a linear function with length of a translation vector and begin to understand that addition and translation both result in shifts, in one and two dimensions respectively. Similarly, students can connect dilation (in which the two variables move in the same direction but with different relative speeds) to multiplication by m in $y = mx + b$, which causes the values of x and y to change at different relative rates. (This relationship may cause students to wonder whether the comparison holds up even when the value of m and the dilation scale factor are negative, giving rise to an excellent opportunity for students to formulate and answer their own questions.)

It's also important for students to connect the congruence of the traces and the figures in this activity to their previous experiences with these concepts. By the time students complete this group of activities, they should recognize that applying reflection, rotation, or translation to a given shape results in a congruent shape, while applying dilation results in a similar shape.

Launch

Expect to spend about 15 minutes.

Tell students that today they will explore three function families by creating and manipulating their own constructions. Explain that students will create and work with the functions to explore the similarities and differences of the function families. Explain further that they will also use the functions they create to make interesting shapes. Tell them that they'll begin by reviewing the reflection function.

Project the sketch **Family Relationships Introduction.gsp.** Explain to students that they will be responsible for planning and carrying out their own investigations, though they can use several resources, including this Introduction sketch.

Have a student demonstrator go to the Skills page and illustrate the first skill listed there: creating an angle parameter θ. Tell students that they can use the other hints on this page to help them if they get stuck on a particular construction.

Call students' attention to the last item on the Skills page about using the Help menu, and particularly the Sketchpad Tips (**Help | Using Sketchpad | Sketchpad Tips**) and the Reference Manual (**Help | Reference Manual**).

Explain, "*In this activity you will explore several families of functions: rotation, dilation, and translation. For each family you should figure out how the members of that family are similar to each other, and how they are different.*"

Have a student demonstrator go to the Sample Responses page. Explain, "*As you work, you should record your observations, and your answers to the questions.*" Tell students how you want them to record their answers (either in the sketch for electronic submission, on their own notebook paper, or on the answer sheets you've passed out). Continue to explain, "*When you answer the questions, use complete sentences just like* *this student did. That will make it easy for you to present your work, and for others to see how you reached your conclusions. Notice that the student used Hide/Show buttons and Animation buttons to help show her work."*

Say, "*This student was exploring reflection, which you've already done. Let's see how the student answered the first question after she constructed the reflection.*" Tell the student demonstrator to drag *x* up and down. Ask, "*For question 1, let's see what happens when* x *is dragged down. Which way does the dependent variable move?*" Elicit student observations. Then say, "*Now drag* x *sideways along the bottom of the domain. Which way does the dependent variable move now?*" Connect students' responses to the step 1 observations recorded in the sketch. Say, "*Notice that this student wrote a clear answer in complete sentences.*"

Tell the student demonstrator go to the Characteristics page. Explain, "*Finally, let's look at how this student summarized the Reflection Family on her paper. What did she write for the Family Name?*" Elicit student responses. Ask, "*What abbreviation did she write for the function, and what does her abbreviation mean?*" Pointing to the r_j, elicit student responses, which should include "reflect across mirror *j*." Ask "*What did she name her independent variable? How did she abbreviate her dependent variable?*" Elicit student responses, settling on "$r_j(x)$ means the reflection across *j* of *x*." Check for understanding of this notation. If appropriate, return to the Sample Responses page to concretely illustrate the meaning.

Continuing on the characteristics page, explain, "*Now let's look at how she described the function's behavior. How did she describe what the function does?*" (It reflects across mirror *j*.) "*What did she say about the relative speed of the variables? Does this match what we just saw?*" (The speeds are the same, as we saw.) "*What about the relative direction of the variables?*" (The direction depends on whether *x* is moving parallel to the mirror, perpendicular, or in some other direction.) "*And what fixed points did she find?*" (When you reflect points that are already on the mirror, they stay right where they are.)

Explain, "*You're going to work in pairs to investigate other important function families: rotation, dilation, and translation. For each family, be sure to write your answer for the questions, to complete a Characteristics Table, and to have an animated sketch that you can use to present your conclusions.*"

Explore

Expect students at computers to spend about 60 minutes.

Assign student pairs to computers and distribute the worksheet. Tell students that the less-experienced partner of each pair should operate the mouse, and that the other partner should record answers and provide coaching, without touching the mouse.

Tell pairs to agree on their answer to each question, and to record their answers in the sketch or on separate paper as you determine. Tell pairs to switch control of the mouse and keyboard after they finish steps 1–4 of the worksheet for each function.

Circulate as students work. Students may need prompting questions to encourage them to move the independent variable in ways that reveal behavior of interest. Note questions and difficulties that arise, so that you can address them during the concluding whole-class discussion.

In a network folder or on a flash drive, collect several sketches that illustrate interesting difficulties students had or interesting observations that they made. As you collect sketches, consider the most logical order in which to show them.

Students who finish early should be encouraged to do the Explore More section in order to create and investigate the Glide Reflection transformation.

Summary

Expect to spend about 15 minutes.

Gather the class. Students should have their worksheets with them. Give students the opportunity to discuss difficulties or misconceptions.

Display particular sketches that you collected to illustrate important elements for the discussion. Validate both correct and incorrect efforts by emphasizing the importance of making mistakes, contemplating them, and persevering in order to learn from them.

Ask students to discuss the comparison chart, and ask them to explain how their observations about covariation (the way in which the independent and dependent variables move in relation to each other) serve to distinguish one function family from another. Try to make sure that all students understand and can apply this idea.

Summarize the important concepts covered: Varying the independent variable to observe the behavior of the dependent variable; labeling the variables with function notation to describe their relationship; paying attention to the relative motion (covariation) of the two variables and to their fixed points; restricting the domain of the independent variable and tracing out the corresponding range; and comparing the shape of the restricted domain to the shape of the range. Link the discussion of the shapes to the concepts of congruence and similarity; this is also an opportunity to ask students "*What transformations produce congruent figures? What transformations produce similar figures?*" This discussion can help students form a clearer concept of congruence and similarity, and to relate the correspondence and covariation views of a function.

Assess

As students leave, ask them to fill out an exit ticket describing one interesting thing they learned, and one thing that they're not sure about.

Review

At the beginning of the next class session, project the sketch **Family Relationships Review.gsp**. Pages 1 through 4 make it easy to summarize the steps of this activity for each family. Each page combines bullet points and examples to summarize the relative rate of change of the variables, the existence and meaning of fixed points, the relationship between the shape and location of the domain and range, and the connection of these shapes to congruence and/or similarity.

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 The relative rate of change and role of fixed points for each function are summarized in the Function Family Characteristics tables below.

Q2 Drawings will vary, but should be roughly consistent with the domain.

Q3 The second try at predicting the range should be an improvement over the first.

Q4 For reflection, rotation, and translation, the range is congruent to the domain, but flipped, rotated, or translated. For dilation, the range is similar to the domain, and either smaller or larger depending on the scale factor.

Q5 When speeds are the same, the range is congruent to the domain; when speeds are different, the range is similar to the domain, either larger (faster) or smaller (slower). When directions are different, the domain is turned or flipped relative to the domain.

Q6 For the same domain, the range for two different functions of the same family has the same fundamental characteristics; only the details change. These details include the direction and distance of the flip for reflection, the center and the angle for rotation, the center and the scale factor for dilation, and the distance and direction of the slide for translation.

Q7 The observations for this question are recorded in the Function Family Characteristics for this function.

Q8 Drawings will vary, but should be consistent with the segment used as the mirror and vector.

Here are possible sample entries in the Function Family Characteristics table:

Family:	**Reflection**		
Notation:	**Function**	**Independent variable**	**Dependent variable**
	r_j	x	$r_j(x)$
Description:	Reflect across mirror j		
Relative Speed:	The variables move at the same speed.		
Relative Direction:	When x moves parallel to the mirror, $r_j(x)$ moves in the same direction. When x moves perpendicular to the mirror, $r_j(x)$ moves in the opposite direction.		
Fixed Points:	The fixed points are the points on the mirror.		
Shape:	The restricted domain and the corresponding range are the same shape and same size (congruent) and flipped.		

Family:	**Rotation**
Notation:	**Function** $R_{C,90°}$ **Independent variable** y **Dependent variable** $R_{C,90°}(y)$
Description:	Rotate about center point C by 90°.
Relative Speed:	The variables move at the same speed.
Relative Direction:	When y moves up, $R_{C,90°}(y)$ moves left; when y moves left, $R_{C,90°}(y)$ moves down. $R_{C,90°}(y)$ always moves 90° clockwise from the movement of y.
Fixed Points:	The only fixed point is the center point C.
Shape:	The restricted domain and the corresponding range are the same shape and same size (congruent), but turned.

Family:	**Dilation**
Notation:	**Function** $D_{C,0.5}$ **Independent variable** p **Dependent variable** $D_{C,0.5}(p)$
Description:	Dilate about center point C by 0.5.
Relative Speed:	The dependent variable moves at half the speed of the independent variable.
Relative Direction:	The two variables always move in the same direction.
Fixed Points:	The only fixed point is the center point C.
Shape:	The restricted domain and the corresponding range are the same shape but not the same size. The range is stretched or shrunk according to the value of the dilation scale.

Family:	**Translation**		
Notation:	**Function**	**Independent variable**	**Dependent variable**
	T_{AB}	w	$T_{AB}(w)$
Description:	Translate by vector *AB* from *A* to *B*.		
Relative Speed:	The variables move at the same speed.		
Relative Direction:	The variables move in the same direction.		
Fixed Points:	There are no fixed points.		
Shape:	The restricted domain and the corresponding range are the same shape and size (congruent). The range is shifted from the domain.		

Family:	**Glide Reflection (from the Explore More)**		
Notation:	**Function**	**Independent variable**	**Dependent variable**
	G_{AB}	z	$G_{AB}(z)$
Description:	Glide reflect using the segment from *A* to *B*.		
Relative Speed:	The dependent variable moves at the same speed.		
Relative Direction:	When the independent variable moves in the direction of the vector, so does the dependent variable. But when the independent variable moves perpendicular to the direction of the vector, the dependent variable moves in the opposite direction.		
Fixed Points:	There are no fixed points.		
Shape:	The restricted domain and the corresponding range are the same shape and size (congruent). The range is both flipped and translated based on the mirror/vector.		

Related Activities:

- *ID the Suspects—Identify Functions (Recommended)*
- *Family Resemblances—Identify Function Families (Recommended)*
- *Reflection Challenges—The Reflection Family (Prerequisite)*
- *Rotation Challenges—The Rotation Family (Recommended)*
- *Dilation Challenges—The Dilation Family*
- *Family Relationships—Rotation, Translation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*

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