Notes for the Teacher

Description:

Students explore dilations as functions. They construct a dilation function, label the variables using function notation, and trace and compare the paths of the variables. They restrict the domain to reveal the corresponding range, and they change the scale and move the center to create different members of the dilation family. Finally, students solve dilation challenges by figuring out the center and scale factor to match an existing dilation.

Note that the material in this activity is also covered in the activity *Family Relationships—Rotation, Dilation, and Translation Families*. This activity contains more explicit student directions (following the pattern established in the prerequisite activity on the reflection family) and covers only the dilation family. The *Family Relationships* activity is more ambitious, expecting students to work independently as they explore and compare the rotation, dilation, and translation families. You should use that activity if your students are prepared for its more independent approach.

This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Construct an independent variable point *x*, a center point, a scale factor, and the dilated image of *x* defined by the center and scale factor.
- Label the dependent variable using function notation.
- Drag the independent variable while tracing both variables, and describe both the variables' relative motion (their *covariation*) and the relationship between their traces (their *correspondence*).

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Change the scale factor to form and investigate different members of the dilation family.
- Restrict the domain of the independent variable to the border of a polygon, and observe and describe the resulting range by dragging the independent variable.
- Identify fixed points of the function, and explicitly compare the relative motion (both speed and direction) of the independent and dependent variables.
- Solve challenges that involve finding a hidden center and unknown scale factor, given an object and its dilated image.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

- *Reflection Challenges—The Reflection Family* (Prerequisite)
- *ID the Suspects—Identify Functions* (Recommended)
- *Family Resemblances—Identify Function Families* (Recommended)
- *Rotation Challenges—The Rotation Family* (Recommended)

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together in developing students' conceptual understanding of functions.

High Cognitive Demand: This activity provides several tasks for which there is no cut-and-dried procedure for students to follow. Though the worksheet provides fairly explicit directions to help students perform the initial construction, the questions it asks require experimentation, inquiry, and analysis.

Mathematical Habits of Mind, Reasoning and Sense Making: Students construct and investigate mathematical objects, and are challenged to answer questions that require tinkering and analysis to understand the behavior of these objects.

Inquiry: The body of the activity supports student inquiry, and the worksheet and challenges contain probing questions that require students to manipulate, observe, and analyze. The probing questions include: What happens when you drag the independent variable to the center point? What if you make the scale factor negative? What's the relationship between similarity and congruence?

Cooperative Learning: Students work in pairs during the exploration portion of the activity, and exchange roles between driving (using the mouse and keyboard) and coaching/recording. Expect students to work purposefully in pairs, to coach each other, and to discuss every part of the activity with their partners. The members of each pair should share the construction and manipulation steps equally, so that each student is fully involved in both creating the function and working with it.

Assessment: You should engage in formative assessment by visiting and questioning student pairs during the exploration phase. You can use the summary discussion to elicit students' understandings, confusions, and questions. The last page of the worksheet is an exit ticket.

Differentiation: The worksheet comes in a long form (with more explicit instructions) and a short form (for students with more experience with Sketchpad). It also includes an optional Answer sheet on which students can write their answers. The concluding Dilation Challenges entail varying levels of difficulty, and are very useful for furthering students' understanding of the Reflection Family and of function concepts in general.

Questioning and Discourse: Most discourse will take place between team members during inquiry, so it's important to encourage team members to describe their observations to each other and to discuss their answers to the questions. The questions on the worksheet, along with teacher observations of the work of different teams, should guide the summary discussion. Since the worksheet includes explicit directions for completing the construction, it's important to ask questions that focus students' thinking on the big ideas, both while they are working in pairs and during the whole-class discussion.

Instructional Strategies: By varying *x*, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

The time students will require for this activity can vary significantly depending on their mathematical background and Sketchpad experience. Because the concluding class discussion is critical, you should be prepared to postpone the challenges contained in **Dilation Challenges.gsp** to another day. While you can assign the Dilation

Challenges as homework exercises, it's preferable to have students work cooperatively in class while you observe their investigations.

Depending on available time and your students' proficiency with Sketchpad, it may be possible for them to do both this activity and the Translation Challenges activity on the same day, and then do the challenges related to these two activities on a second day.

The three stages of the activity are described in more detail below, in the sections on Launch, Inquiry, and Summary.

Worksheet: The worksheet comes in two forms, both of which contain the same questions. The longer form is useful for students who are relatively new to Sketchpad, as it provides explicit steps to follow. If students have experience with Sketchpad, the shorter form is preferable, as it expects more initiative from students and it concentrates more on the mathematics to be explored and less on technicalities. (Giving students unnecessarily explicit directions risks miring students in details, discouraging their thoughtfulness and creativity.) Some teachers may choose to provide one short form and one long form to each pair of students, telling them to work from the short form, referring to the details in the long form as needed.

Answer Sheet: Many teachers prefer their students to write their answers on their own paper. Alternatively, you might ask students to submit their answers electronically, or you can use the provided answer sheet. If you assign electronic submission, tell students that they can include screen captures by choosing **Edit | Select All** and then **Edit | Copy** in Sketchpad. They can paste the resulting image into a word processor, email, or other document. (The image includes traces and is cropped to the window border, so that students can easily control the area being copied and pasted.)

Transfer of Learning: To get the most value from this activity, students must connect what they've learned to other representations of functions. By dragging the independent variable *x* in this activity, they can relate its smooth motion to the continuous variation of a numeric variable. Moreover, comparing the relative speed and direction of the independent and dependent variables can help students think about the relative rate of change of the variables of a linear function. Dilation provides an excellent opportunity for connecting Geometric Functions to linear functions, since both dilation and multiplication are scaling operations. It's also important to connect the similarity of figures in this activity to students' previous experiences with similarity. (Challenge 6 specifically addresses similarity.)

Launch

Expect to spend about 5 minutes.

Tell students that today they will explore the dilation function family by creating and manipulating similar functions. They will then investigate the similarities and differences, and use the functions they create to make interesting shapes.

Remind students to use the Help menu if they have difficulties, and tell them that in addition to the Reference Center they may find the Sketchpad Tip on Rotating and Dilating useful. (Choose **Help | Using Sketchpad | Sketchpad Tips | Transform | Rotating and Dilating.**)

Explore

Expect students at computers to spend about 25 minutes.

Assign student pairs to computers and distribute the worksheet. Tell students that the less-experienced partner of each pair should operate the mouse first, and that the other partner should provide coaching and record observations without touching the mouse. Tell students to switch between their roles as operator and coach/recorder after they answer Q4 on the worksheet.

Tell pairs to agree on their answer to each question, and to record their answers in the sketch or on separate paper as you determine.

Circulate as students work, asking them what they notice as they drag the independent variable and observe the dependent variable. Encourage them to drag in various directions and at various speeds as they look for patterns in the relative rate of change of the variables. Make note of questions and difficulties that arise, and encourage students to raise these questions or difficulties during the concluding whole-class discussion.

Whenever possible ask questions instead of providing information. Pay particular attention to students' observations for Q1 ("What happens when you drag point *x* up?" or "How would you describe the difference between the two directions?") and to their explanations for Q5 and Q7 ("How does the scale relate to the relative speed of the variables? What would happen if *scale* = 5?"). As you check on the groups, plan the order in which to call on students to bring out aspects of function reasoning in a logical way. This can ensure that simple observations precede more sophisticated explanations and interpretations. Also, have selected pairs recall those questions or difficulties that would be beneficial for the entire class to discuss.

If time and technology permit, use a network folder or on a flash drive to collect several student sketches that illustrate interesting difficulties or interesting observations. As you collect sketches, consider the most logical order in which to show them. That way, you can lead a discussion about the relative rates of change of the variables and the shapes traced out when the domain is restricted.

Students who finish early should be encouraged to begin **Dilation Challenges.gsp.**

Discuss and Summarize

Expect to spend about 10 minutes.

Gather the class. Students should have their worksheets with them. Give students the opportunity to discuss difficulties or misconceptions.

Summarize the important concepts covered:

- varying the independent variable to observe the behavior of the dependent variable;
- labeling the dependent variables with notation that describes their relationship;
- attending to the relative motion of the two variables and to their fixed point(s);
- restricting the domain of the independent variable;
- tracing out the corresponding range; and
- comparing the shape of the restricted domain to the shape of the range.

Link the discussion of the shapes to the concepts of congruence and similarity; this is an opportunity for students to describe in their own words what it means for two figures to have exactly the same shape but different sizes, and to relate the term *similarity* to their own observations.

Based on the order you determined while circulating, call on students to discuss their answers for Q1, Q5, and Q7, and to compare the answers to these three questions. Use these particular students' contributions to try to generate a lively and probing discussion, and through the discussion to develop a class consensus on the relationships between the scale slider, the relative speed of the variables, and the ratios of corresponding sides.

The discussion should also touch on particularly interesting values of the scale. What do students predict will happen when $scale = 1$, $scale = 0$, or $scale < 0$? These questions are a good lead-in to the connection to linear functions. You can ask students to give an example of a linear function (expressed as $y = mx + b$) for which the value of y increases half as fast as the value of x, an example for which the value of y increases twice as fast as the value of x, and an example for which the value of y decreases at the same speed as the value of x increases.

Throughout the discussion, validate both correct and incorrect efforts by emphasizing the value of making mistakes, contemplating them, and persevering in order to learn from them.

Assess

Just before students leave, ask them to fill out an exit ticket describing one important thing they learned and one thing that they're not sure about.

Class discussion during the subsequent review, along with student work on the Dilation Challenges, are important opportunities for assessment.

Review and Challenges

It's important to devote part of another class period to reviewing what students have learned so far and to consolidating that learning by means of the Dilation Challenges. (It's best to do this during class time rather than for homework so that you can monitor and discuss students' work and assess their understanding.)

Expect to spend about 30 minutes.

Project the sketch **Dilation Challenges Review.gsp**. Pages 1 through 4 list the various objectives of this activity, to facilitate a brief review that will help students' retention. On each page have a different student volunteer press the bullet(s) to reveal activity objectives, and then use the page to illustrate that page's objectives.
Page 1: Ask the student to press each phrase in italics to reveal in turn the independent variable, the center point, the scale parameter, the dependent variable, and the function notation. Then have the student drag point *x* while observing the relative motions of the variables.
Page 2: After showing the first bullet, have the student drag *x* first vertically and then horizontally while observing the relative rate of change for both motions. After the second bullet, ask the student to press the italicized phrase *Move the center*, creating a different member of the same function family. Have the student drag *x* again, and observe the relative motion of the dependent variable. Ask the student to press the italicized phrase *change scale factor s*, creating yet another member of the same function family. Have the student drag *x* again, and observe the corresponding motion of the dependent variable.
Page 3: Show the first bullet and have the student drag *x* to find a fixed point, and to verify that there's only one fixed point. Show the second bullet and have the student merge the independent variable *x* to the polygon and then drag *x* to show the effect of a restricted domain.
Page 4: This page introduces the dilation challenges from **Dilation Challenges.gsp**. Explain that one of the challenges involves finding a hidden center and scale factor that connects the red polygon to the blue one. Having a volunteer do these steps:

1. Drag the independent variable *x*.
2. Restrict *x* to the red polygon.
3. Erase traces and animate *x*.
4. Drag *C* to adjust the center, and edit *s* to change the scale.
5. Erase traces and animate again.

Before the volunteer determines the center or scale correctly, stop the demonstration and tell students that discussing and solving these puzzles will develop their intuitive understanding of the dilation family. Send students to the computers in pairs, have them open **Dilation Challenges.gsp,** and ask them to solve as many of the puzzles as they can.

One outcome of working on the puzzles is the ability to look at two dilated shapes and quickly locate the center and approximate the scale. Students' strategy for finding the center is likely to involve mentally shrinking the larger polygon through the smaller polygon and visualizing where it would shrink to a point—the fixed point (or center) of the dilation. Some students may realize that any line defined by corresponding points on the polygons passes through the center, and that two such lines suffice to locate the center precisely. Some students may also realize that they can measure the distance from the center point to corresponding points on the two polygons to find the ratio. (Avoid explaining these strategies; both of these realizations are much more valuable if students discover them for themselves.)

[In a later activity, *Compose a Locus*, students will learn how to construct the range corresponding to a restricted domain. This technique makes it easier to solve puzzles such as these, but would be counter-productive if introduced now.]

Challenges 6 and 7 merit particular attention during the concluding class discussion, in part because their similarity to Challenges 6 and 7 from **Rotation Challenges.gsp** can inform a comparison between rotation and dilation.

Challenge 6 also makes the connection between dilation and similarity, and provides an opportunity to develop this connection by building on the discussion of congruence from the Rotation Challenge activity.

Challenge 7 helps prepare students for the inverse function concept, which they will encounter explicitly in a later activity. This challenge has two polygons (P_1 and P_2) but does not identify which is the restricted domain and which is the range. The challenge asks students to find the center and scale that dilates P_1 to P_2, and then to find the center and scale that dilates P_2 to P_1. Thus the challenge is to find a function that will take the range back to the corresponding domain—in other words, to find the inverse function. (Don't introduce the terms *inverse* or *inverse function* yet. Each transformation challenge activity has a similar question; after answering these questions for several different function families, students will be well-prepared to formalize and name the concept at a later time.)

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 Students should observe that $D_{C,0.5}(x)$ always moves in the same direction as the independent variable x, but moves half as fast and thus half as far. (This connection between the speed of motion and the size of the traces may not be immediately apparent to all students; they may have to think about how the speed of the variable determines the size of the pattern it traces.)

Q2 Answers will vary, and may include observations like these: The shapes look the same, but are of different sizes. The one made by $D_{C,0.5}(x)$ is smaller than the one made by x. It's also closer to the center point C. Horizontal parts of one shape are also horizontal on the other shape. Students may realize that the trace of $D_{C,0.5}(x)$ is half the size and half the distance from C as the trace of x, thus connecting the size and location with the scale factor.

Q3 When you drag the independent variable through the center point, the independent and dependent variables meet at the center. This happens because when x is at the center, a point half as far away is also at the center. (Numerically, half of zero is still zero.) The two traced shapes are still the same shape, with the dependent variable's trace half the size of the independent variable's trace.

Q4 The independent variable is described as an independent object. The dependent variable is described as the dilation of point x by 50% about center point C.

Q5 When you drag x up, $D_{C,s}(x)$ also moves up, but faster. When you drag x left, $D_{C,s}(x)$ also moves left, but faster. Students may observe that the dependent variable moves twice as fast as the independent variable, or that it moves twice as far from the center. (This question provides a second opportunity for students to connect the relative rate of change of the two variables with the shapes that they trace out.)

Q6 The traces have the same shape and are at the same angle, but are different sizes. The one made by $D_{C,s}(x)$ is larger than the one made by x. It's also farther from the center point C. Students may observe that the shape made by $D_{C,s}(x)$ appears to be twice as large as that made by x.

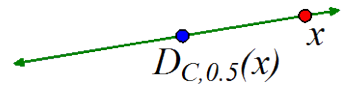
Q7 When $s = 0.3$, the shapes are still similar, but the trace made by $D_{C,s}(x)$ appears to be about 1/3 the size of the trace made by x. Many students will conjecture that the trace made by $D_{C,s}(x)$ is exactly 3/10 the size of the trace made by x.

Q8 You can drag x to bring the two variables together; this happens at point C, the center of dilation.

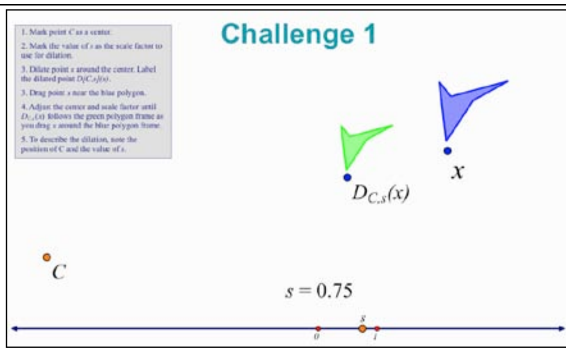
Q9 The independent variable is restricted to the border of the polygon. It always stays along the border.

Q10 The range has exactly the same shape as the restricted domain, but is larger or smaller depending on the scale factor.

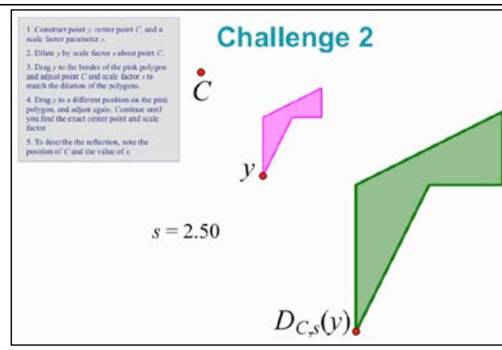
Q11 Answers will vary. If you have a single pair of points, you can draw the line that contains both points, and drag the independent variable along the line. The points will meet at the center of dilation. Alternatively, you can turn on tracing for the line and vary *x;* the traces will all pass through the center point. If you have two shapes, you can draw a line through one pair of corresponding points and another line through a different pair of corresponding points. The intersection of the two lines is the center of dilation. If you measure the distance from this center point to each of the two points on the same line, the scale factor is the ratio of those distances.



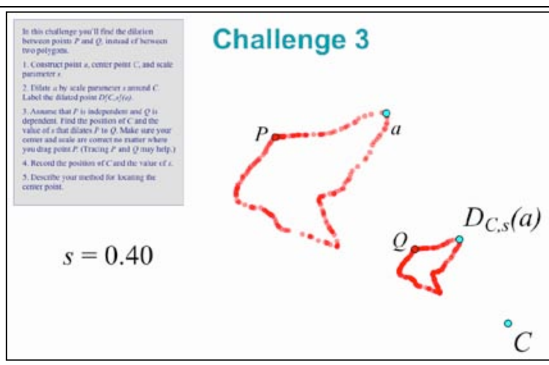
Dilation Challenges



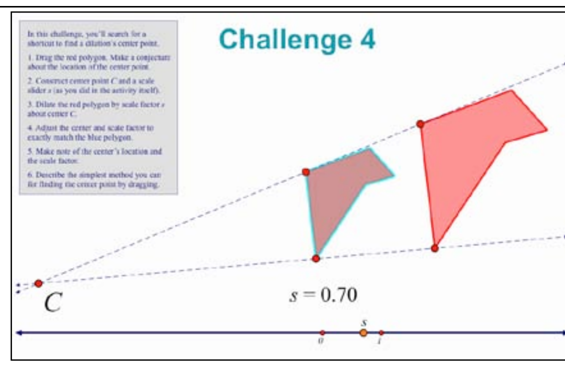
Students will likely approach this problem using guess-and-check methods.



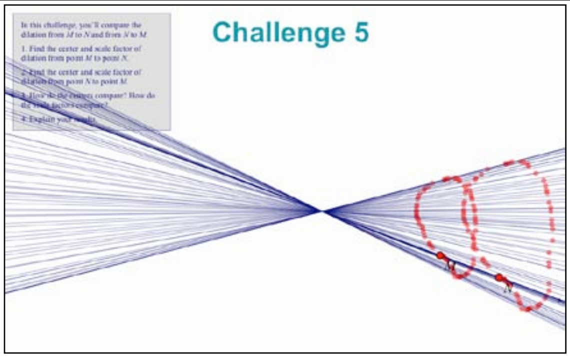
Students may begin to search for a more efficient method than guess-and-check.



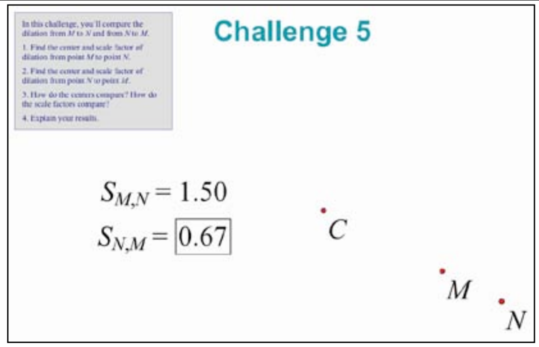
This student has used traces, because a single pair of values does not provide enough information to find the center of dilation and scale factor. It's also possible to find the fixed point by dragging.



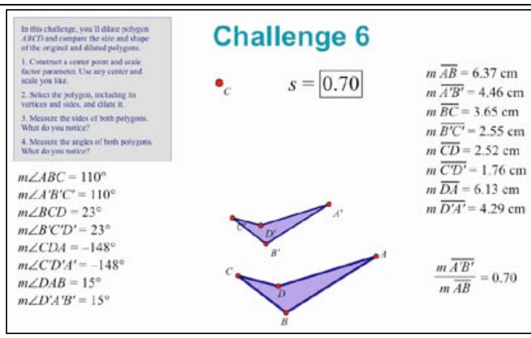
This student constructed lines through corresponding points of the two polygons to find the center. She then adjusted the scale to make the new dilated polygon exactly cover the blue one.



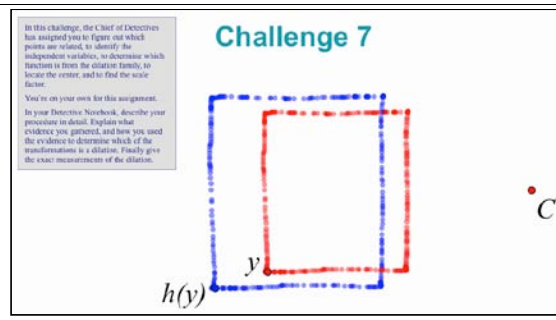
This student constructed a line through *M* and *N* and turned on tracing for the line. The trace of the line reveals the center of dilation.



The scale factor from *M* to *N* is 1.50, and the scale factor from *N* to *M* is 0.67. They are multiplicative inverses. This challenge suggests the concept of inverse functions, which will be addressed in a later activity.



Corresponding angles are equal. The ratio of two of the corresponding sides (*AB* and *A'B'*) is equal to the scale factor. What will happen if we check the other ratios? This challenge can inform a discussion of similarity.



Function *h* is the only dilation among the four functions.

Related Activities:

- *ID the Suspects—Identify Functions (Recommended)*
- *Family Resemblances—Identify Function Families (Recommended)*
- *Reflection Challenges—The Reflection Family (Prerequisite)*
- *Rotation Challenges—The Rotation Family (Recommended)*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Dilation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*

License (CC-BY-NC-SA 3.0)

Portions of this material are based upon work supported by the National Science Foundation under award number DRL-0918733. Any opinions, findings, and conclusions or recommendations expressed in this work are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.