Notes for the Teacher

This activity introduces the locus construction to create the range of a function with a restricted domain, enabling students to easily investigate function behavior by changing the parameters that define the function. For instance, students can change the angle of a rotation while observing the effect on the range. Students experience functions as mappings that take a set of input points (the restricted domain) to a set of output points (the corresponding range, constructed as a locus).

This activity helps to advance students' thinking about function in two ways. Functions appear more explicitly in this activity as mappings that take an entire set of input points to an entire set of output points. At the same time, functions are used by students as objects in their own right, objects that can be operated upon by combining them, editing them, and animating them.

Function as Mapping: The initial view of function is that of an action that takes a single input and produces a single output. One might term this an *atomic* view: a function takes a single geometric point or numeric value (the input atom) and produces a single point or value (the output atom). As students gain in understanding, they begin to imagine the function operating on many input points at once, thus producing many output points simultaneously. Students have already analyzed the traces left by input and output points as they vary, both with and without restricted domains that limit how the independent variable can vary. However, these traces were produced over time, one input/output pair at a time. In this activity students use an entire set of input values (the values belonging to the restricted domain) to construct an entire set of output values (the locus of the dependent variable) in a single step. Thus the locus construction introduces a *collective* view: the function transforms an entire set of input values to create a corresponding set of output values. One consequence is that students can change the definition of a function (by editing or animating an angle or a scale factor) and immediately see the change in the mapping. There's no longer a need to explicitly vary the independent variable over time. [This significant transition in students' thinking is not often addressed explicitly in curricula or textbooks; we do students a disservice if we assume it's an easy step and take it for granted.]

Function as Object: A related transition in students' thinking is from considering a function as an action or process to considering a function as an object. Students already have some experience with editing functions (for instance by dragging the mirror or changing the angle of rotation) in order to modify their effects, and they have compared families of functions that result from such modifications. Composition of functions adds another push in this direction, by combining two functions to form a single function. In this activity students take this process one step further, by animating the angle of a rotation function and the scale of a dilation function. By observing how the mapping changes, students accept functions not just as actions or procedures, but as *mathematical objects* that can be combined with other objects and that can operated upon in various ways.

In one sense, this activity consists of variations on concepts previously learned. Using action buttons to animate parameters is just an extension of changing their values by other means, and the locus of the dependent variable is an extension of tracing the path of the dependent variable. Yet these steps automate important mathematical elements of students' experience with functions even as they turn *range* into an object that students can construct and interact with. Other elements of the activity solidify students' understanding of composition, relative rate of change, and function notation.

Expect the activity to take a single class period (about 45 minutes).

> This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Create a scale factor and center point to define a dilation function.
- Create an angle measurement and center point to define a rotation function.
- Describe both functions using function notation.
- Compose two functions and label the dependent variable using function notation.
- Describe the relative rate of change of the composed function's variables.
- Restrict the function's domain to the border of a polygon.
- Construct the range corresponding to the restricted domain as a locus.
- Animate the independent variable over the restricted domain to verify that the locus construction includes all corresponding values of the dependent variable.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Modify the scale factor and angle measurement to observe and analyze changes in the mapping from domain to range.
- Animate the scale factor and angle measurement to observe and analyze a continuously changing mapping as the function itself is transformed through a subset of its function family.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

Before undertaking this activity, students must already be familiar with function composition by doing the following activity:

- *Transform Twice—Function Composition* (Required)

Students must also have completed one or the other of the following:

- Three of the four function challenge activities (*Reflection Challenges, Rotation Challenges, Dilation Challenges, Translation Challenges),* or
- *Family Relationships—Rotation, Dilation, and Translation Families.*

These prior activities are also highly recommended:

- *ID the Suspects—Identify Functions*
- *Family Resemblances—Identify Function Families*
- *Dance the Dependent Variable—Geometric Function Dances*

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to develop students' conceptual understanding of functions.

Cognitive Demand: The short form of this activity contains high level directions and questions that require student initiative, experimentation, and analysis. The questions on both forms concentrate on the connections between students' sensory-motor experiences and the mathematical objects they are manipulating and observing.

Mathematical Habits of Mind, Reasoning and Sense Making: Students build all of the mathematical objects and then explore and analyze the behavior of these objects. The short form worksheet encourages students to be particularly self-reliant. Students construct the locus and make sense of it as the range corresponding to the restricted domain. In that process students come to view the function as a mapping from the domain to the range.

Inquiry: The worksheet contains probing questions that require students to manipulate, observe, and analyze. As students work, they will pose and answer questions, such as "What happens when I combine two functions, using the output of one as the input to another? Do the domain and range exhibit similarity or congruence?" The Explore More question expects students to create their own experiment and report on the results.

Cooperative Learning: Students work in pairs throughout the activity. Expect students to work purposefully in groups, to coach each other, and to discuss every part of the activity with their partners.

Assessment: You should assess student understanding by visiting and questioning student pairs, not only observing their work but also encouraging and guiding them. Use the summary discussion to probe for students' understandings and confusions. The last page of the worksheet is an exit ticket.

Differentiation: Different levels of students are supported by the worksheet, available in a long form (containing detailed Sketchpad instructions) and a short form (containing only a mathematical overview of the objects to be constructed). Student pairs work together, coaching each other and consulting with other pairs for additional support. The optional Explore More question is designed to engage students with different levels of background knowledge and to encourage self-directed work.

Questioning and Discourse: Since most discourse will take place between and among pairs during inquiry, it's important to encourage students to describe and explain their construction methods and their observations to each other. Use the summary discussion to focus students' thinking on the big ideas, and the role that mathematical ideas play in the activity. These ideas include functions as mappings from domain to range, and functions as objects that can be transformed through a continuum of members of a function family.

Instructional Strategies: By varying the independent variable, and later animating parameters to transform functions themselves through a set of related family members, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

Prepare by printing enough copies of the worksheet (**Compose a Locus Worksheet.pdf**) for your class. Note that the worksheet is available in long and short forms; the forms contain the same questions, but the long form contains more detailed instructions leading up to the questions. The short form is on a higher cognitive level, and gives students more responsibility for figuring out the details. It concentrates on the mathematical objects to be constructed and manipulated, omitting previously experienced construction details. Students who have already constructed and investigated functions from several families should be ready to use the short form, though they may need to use the Help menu (**Help | Reference Center**) to remind themselves of specific techniques. Consider passing out the short form to all the teams and printing a few copies of the long form for students to circulate as needed for the additional detail.

Review the two sketches that accompany the activity. Thought students do their work in a new sketch, the prepared sketches can be useful for other purposes:

Compose a Locus Work.gsp shows a typical construction at various stages identified by the numbered steps of the long form of the worksheet, and may be useful when you discuss and summarize with the class.

Compose a Locus Challenges.gsp presents three challenges involving function composition. These challenges are beyond the scope of the activity itself but can serve as extra credit or as a follow-up activity for an advanced class. (These challenges are numbered 4, 5, and 6, extending the numbering of the challenges in the activity *Transform Twice—Function Composition*.)

Launch

Expect to spend about 5 minutes.

Explain to students that in today's activity they will continue working with composition of functions, and that they'll use two constructions they may not have used before.

First, they will create an Animation button to animate the independent variable. They'll also create an Animation button to animate a scale factor and an angle and observe the effect of a function's behavior.

Second, they will construct a locus of the dependent variable after they've restricted the domain of the independent variable. This is far better than a trace because it's always onscreen, because its shape and size can easily be compare to the restricted domain, and because (unlike traces) it changes immediately as the function changes.

Pass out the worksheets to the class. If you're using the short form of the worksheet, tell students that they will be expected to remember some construction details from previous experience, to consult with their partners and with other pairs when they need help, and to refer to a copy of the long form or to the Reference Center when they are stuck.

Remind students that each pair will have an operator (using the keyboard and mouse) and a coach/recorder (making suggestions and recording notes). Tell them to switch roles after answering Q3.

Finally, ask students to use this activity to monitor their understanding of new function concepts. While working, students should be thinking, “What do I understand well? What aspects are still unclear or confusing, and how can I clear them up?”

Explore

Expect to spend about 30 minutes.

Circulate as students work, making sure students write clear descriptions in complete sentences of the behavior they observe. Emphasize the importance of noting relative rate of change, fixed points, and other features.

At the beginning of the activity students construct the two functions separately and then use the **Merge** command to compose the functions. Some students may ask why they can’t just construct the dilated image and then rotate that image. The answer is that they can once they understand the mathematical fine points of what they’re doing. This activity’s instructions intentionally direct them to create two separate functions before composing them, precisely to emphasize the action of composition. Later, when they fully understand the process, it will be safe to make composition an implicit step that occurs while creating the two functions.

Pay particular attention to the aspects that are new, particularly the locus representation of the entire range. It allows students to modify the function and get immediate feedback on how every point of the domain is transformed by the function, a capability that’s simultaneously powerful and potentially confusing. Make sure that students can explain what that locus means in terms of the function.

Students are told to continue animating the independent variable even as they make changes to the scale factor or rotation angle. The ongoing variation of the independent variable emphasizes how the locus points are actually generated: students see how the domain is the set of all input points to the function and the locus is the set of all output points.

In the Explore More, students animate the scale factor or rotation angle instead of the independent variable. This is a very important shift: students are asked to transition from observing the dynamic nature of a fixed function to observing the changes in the range as the function itself changes. Pay close attention to students' descriptions for Q6 through Q8 to see how they react to this transition.

Students who finish Q8 should undertake the Challenge sketch.

Be sure to call students to a halt early enough to conduct a summary discussion.

Summarize

Expect to spend about 10 minutes.

The first focus of the discussion should be the concept of composition. Did this activity help students to solidify their understanding of composition? Was the process clearer and more meaningful to them than it was in the *Transform Twice—Function Composition* activity?

Ask students what they thought about the transition from viewing the range by tracing to constructing the range as a locus. Did they understand that the locus represents the application of the function *to the entire domain simultaneously?* This important idea of the function acting collectively (on many points at once) instead of atomically (on a single point at a time) should be discussed and reflected upon.

Ask students why they think the instructions called for them to continue the animation of the independent variable, with only one brief pause, throughout the activity. Hopefully some students will observe that the activity is designed to emphasize the role of the variables, and it does so by continually varying them.

Finally, ask students what they thought about animating the angle and scale parameters in Q6 and Q7. Ask how this is different from the button that animates the independent variable. Students should be able to relate their answers to this question to the concept of function families.

To refer to a completed sketch during the summary discussion, you can use a student sketch that you've collected, or you can use **Compose a Locus Work.gsp.** This sketch contains the finished constructions for Q6 and Q7, which can help support the discussion about how the functions themselves are being changed, and how the changing range shows the effects of the modifications to the functions.

The last page of **Compose a Locus Work.gsp** contains bullet points summarizing some of the important ideas from this activity, and may be a useful way to end the discussion.

Consider providing students with the sketch **Compose a Locus Challenges.gsp** as a fun challenge, or as an extra-credit assignment.

Assess

Just before class ends, ask students to fill out the exit ticket. Question 1 can provide useful information about students' understanding of the range as a set of output values. Question 2 is intended to assess students' understanding of how their animation of the parameters actually changed the functions themselves. For example, students should observe that the animation showed the dilation function becoming different members of the dilation function family as the value of *s* changed.

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 This function produces a rotated dilated image. The dilated image is somewhat closer to the center point, but still quite near to x. The dilated image moves a bit more slowly than x, and always in the same direction. The center point C is the fixed point.

Q2 Drawings will vary. Both x and $D_{C,s}(x)$ should be labeled.

Q3 The range is a bit smaller than the domain, and is closer to the center point C. It's also rotated relative to the polygon domain; it looks like it's turned by about 45°.

Q4 Pictures will vary; by this time the range is turned 90° from the domain. The locus represents the range corresponding to the restricted domain.

Q5 Changing s to 0.5 made the shape of the range much smaller, and changing θ to −90° flipped the range to the opposite side of the center point.

Q6 Descriptions will vary: the range moves from being aligned to the domain (at 0°) to being completely opposite to the domain (at 180°).

Q7 Descriptions will vary, but they should address the question of "What happened when s became negative?" In fact, the range changes from twice the size of the domain (when $s = 2$) to a point (when $s = 0$) to an inside-out version of its original shape (a reflection through C when $s < 0$).

Q8 The answer to this question is the construction itself.

Related Activities:

- *ID the Suspects—Identify Functions*

- *Family Resemblances—Identify Function Families*
- *Reflection Challenges—The Reflection Family*
- *Rotation Challenges—The Rotation Family*
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Translation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*
- *Transform Twice—Function Composition*
- *Special Effects—A Swirling Transformation*
- *Animated Special Effects—Swirl a Picture*

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