Notes for the Teacher

Students explore patterns in the decimal representations of fractions. No knowledge of long division is required for this activity.

A color calculator converts fractions less than one to their decimal equivalents, representing each digit in the decimal expansion with a unique color. All 0's are red, all 4's are green, all 5's are blue, etc. Viewing the decimal expansion of a fraction with this color-coding reveals visual patterns in the digits.

Whereas traditional calculators show a limited number of digits to the right of the decimal point, the color calculator shows as many digits as students would like. The ability to view a plethora of digits makes it easy to search for patterns in the decimal representations. The digits are displayed in rows and columns, and students can adjust the number of rows or columns to spotlight repeating patterns.

Students can pose and explore questions such as: "Which fractions, when represented as a decimal, have a number (color) pattern that repeats over and over again?" (repeating decimals) and "Which fractions, when represented as a decimal, end in all zeroes?" (terminating decimals)

Objectives:

- Students will explore the decimal representation of fractions less than one, viewing many digits to the right of the decimal point.

- Students will explore color-coded decimal representations of fractions with each digit replaced by a unique color throughout the expansion.

- Students will investigate patterns in the decimal representations of fractions.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (5) Use appropriate tools strategically; (7) Look for and make use of structure.

Common Core State Content Standards: 4.NF5, 6; 5.NBT1, 2, 3; 5.NF3; 7.NS2

***Grade Range:*** Grades 4–7

Introduce:

Open **Color Calculator**--**Decimal Representations of Fractions.gsp** and distribute the worksheet. Use a projector to show sketch page "Color Calculator." The fraction displayed is $\frac{1}{4}$.
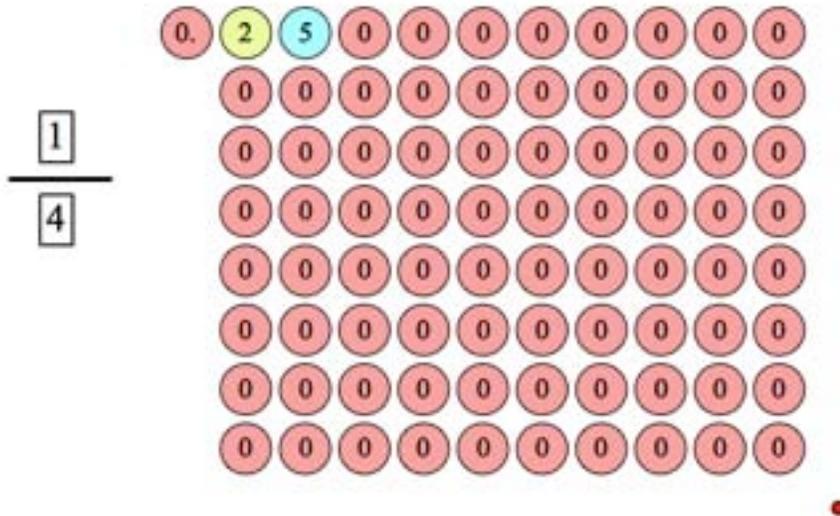
Introduce the color calculator: "This color calculator converts any fraction less than one to its equivalent decimal representation." Ask students to describe what they see on screen. Sample responses include:

- *The calculator shows $\frac{1}{4}$ as a decimal.*

- *One fourth is equal to one quarter, and that's 0.25.*

- *All of the zeroes are red. The 2 and 5 each have a different color.*

- *One fourth is equal to 0.25. There are many zeros to the right, but they don't change its value. No matter how many zeroes there are to the right of the 5, the value of the decimal is still the same.*

Explain that students can change the fraction simply by highlighting its numerator and/or denominator with the **Arrow** tool and entering new numerical values using the computer keyboard. Demonstrate how to change $\frac{1}{4}$ to $\frac{1}{7}$.

$$\frac{\boxed{1}}{\boxed{7}}$$
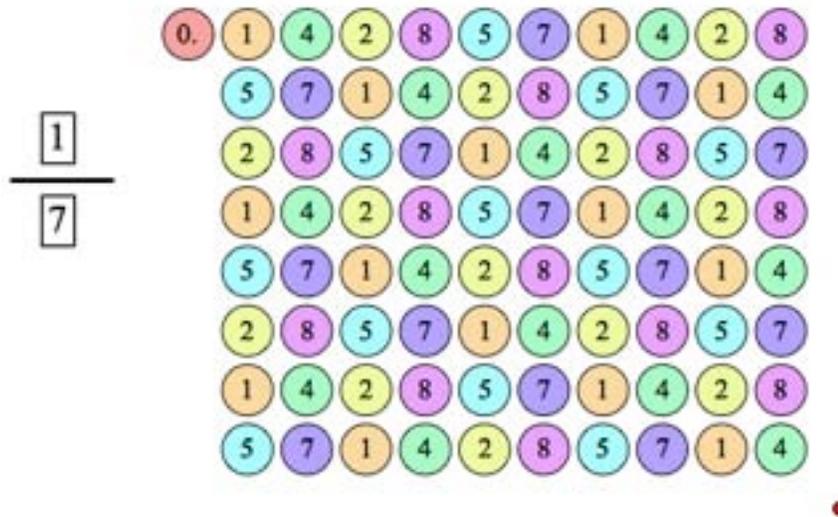
Again, ask students to describe what they see. Sample responses include:

- $\frac{1}{7}$ *has many digits that aren't zero. In fact, none of the digits to the right of the decimal point are zeros.*

- *There are many more digits, and each is a different color!*

- *I see a pattern of digits that repeats over and over: 142857*

Drag the red point around the screen using the **Arrow** tool. As you do, the number of rows and columns in the decimal representation of $\frac{1}{7}$ will change. The color patterns that emerge will surely elicit reactions from your students. Drag until there are 7 decimal digits shown in each row. In the pattern below, students will likely notice that each diagonal consists of a single color.

Drag the red point one place to the left so that the rightmost column of numbers is all 7's. In this arrangement, it is very easy to see that the pattern 142857 keeps repeating.

You can also drag the red point up and to the right so that all the digits in the decimal expansion of $\frac{1}{7}$ appear in a single row:
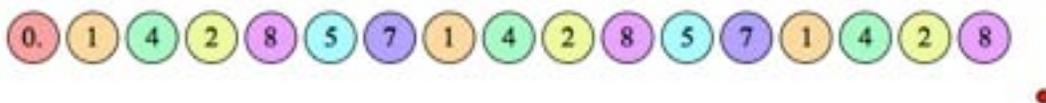
(This activity does not assume that students know the long-division algorithm, but if your students are acquainted with the algorithm, calculate $1 \div 7$ on the board to show that the color calculator does indeed give the decimal value of $\frac{1}{7}$.)

Explain that students will use the color calculator to explore patterns in the decimal representations of fractions. You can leave the activity open ended and ask students to follow their own curiosity and to report their findings. Or, you can write a selection of the following challenges on the board and ask students to think about them as they work:

- Find several fractions whose decimal representations have exactly one non-zero digit to the right of their decimal points. (e.g., 0.6000…) What do these fractions have in common?

- Find several fractions whose decimal representations have exactly two non-zero digits to the right of their decimal points. (e.g., 0.47000…) What do these fractions have in common?

- Find several fractions whose decimal representations consist of just one repeating digit. (e.g., 0.33333…) What do these fractions have in common?

- Find several fractions whose decimal representations eventually "stop" and end in all zeroes. What do these fractions have in common?

- Find several fractions whose decimal representations repeat a sequence of digits over and over again. (e.g., $\frac{1}{7} = 0.142857142857\ldots$)

- Find some fractions whose decimal representations have really long strings of digits that eventually repeat.

- What do you notice about the decimal representations of the fractions $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots, \frac{6}{7}$?

- What do you notice about the decimal representations of fractions with a denominator of 13?

- What do you notice about the decimal representation of the fraction $\frac{1}{81}$?

- Are there any fractions whose decimal representations never repeat or end in zero?

Explore:

Assign students to partners and send them in pairs to the computers. Have students open **Color Calculator--Decimal Representations of Fractions.gsp** and go to page "Color Calculator."

Note that there are two other pages in the Sketchpad document: "Just Digits" and "Just Color." The "Just Digits" page represents decimals the traditional way, with digits only— no colors. The "Just Color" page represents decimals with colors only —no digits.

You might suggest to students that they conduct their investigations on the "Just Color" page. That way, they can focus on the patterns inherent in the decimal representations without worrying about the specific digits. Students who are color blind should use the "Just Digits" page.

As you circulate, watch students as they work. Ask students what they are investigating and what observations they have made. Provide guidance in the form of questions or prompts for students who need help.

If you notice that students are only exploring fractions with a numerator of 1, encourage them to try other numerators as well.

Discuss:

Call students together to discuss and summarize what they have learned. Talk about what fractions students investigated. Open **Color Calculator--Decimal Representations of Fractions.gsp** and go to page "Color Calculator" and let volunteers demonstrate their findings. Introduce the terms *repeating decimal* and *terminating decimal* in the context of the model as they arise.

Student responses may include the following:

- *Any fraction less than 1 that has a denominator of 10, from $\frac{1}{10}$ to $\frac{9}{10}$, has*

  exactly one non-zero digit to the right of the decimal point. That's because $\frac{1}{10}$ =

  0.1, $\frac{2}{10}$ = 0.2, $\frac{3}{10}$ = 0.3, etc.

- *We looked for a fraction whose decimal representation had exactly two non-zero digits to the right of the decimal point. It wasn't hard to find several of*

  these fractions. We knew that $\frac{23}{100}$ = 0.23 and $\frac{59}{100}$ = 0.59, so we thought that

any number from 1 to 99, when divided by 100, would do the trick. But that wasn't quite right: $\frac{10}{100} = \frac{1}{10}$, *and that's equal to 0.1.*

- *We found that denominators of 13 made nice diagonal patterns. When the numerator was 1, 3, 4, 9, 10, or 12, the decimal expansion contained the sequence of digits 076923, but in each case, starting with a different digit. When the numerator was 2, 5, 6, 7, 8, or 11, the decimal expansion contained the sequence 153846, but again, starting with a different digit depending on the numerator.*

- *We started with* $\frac{1}{3}$ *and saw all the circles were green. That meant* $\frac{1}{3}$ *was equal to 0.333.... The 3's continue on and on forever. We then tried to find other fractions that repeated a single digit. We found* $\frac{2}{3}$ *was repeating sixes (0.666...). Then we found* $\frac{1}{9}$ *was repeating ones (0.111....). This always happens for fractions with 9 as a denominator. The numerator of the fraction is the number that repeats in the decimal.*

$$\frac{2}{9} = 0.222... \qquad \frac{3}{9} = 0.333... \qquad \frac{4}{9} = 0.444...$$

$$\frac{5}{9} = 0.555... \qquad \frac{6}{9} = 0.666... \qquad \frac{7}{9} = 0.777...$$

$$\frac{8}{9} = 0.888...$$

- *We couldn't find a fraction that was equal to 0.999...*

- *We decided to try* $\frac{1}{99}$ *to see if any patterns occurred that were similar to the ninths. Sure enough, there were! The numerator repeats as two digits. So,* $\frac{1}{99} = 0.010101...$, $\frac{2}{99} = 0.020202...$, $\frac{13}{99} = 0.131313...$, $\frac{98}{99} = 0.989898....$

- *We investigated fractions with a denominator of 7 because we liked the patterns. We found the same digits repeat in the same sequence, but with a different starting digit:*

$$\frac{1}{7} = 0.142857$$

$$\frac{2}{7} = 0.285714$$

$$\frac{3}{7} = 0.428571$$

$$\frac{4}{7} = 0.571428$$

$$\frac{5}{7} = 0.714285$$

$$\frac{6}{7} = 0.857142$$

- *We looked for fractions whose decimal representations didn't repeat for a long time. We found $\frac{1}{17}$ had 16 digits that repeated, and $\frac{1}{19}$ had 18 digits that repeated. We thought $\frac{1}{21}$ might have 20 digits repeat, but it had only 6 repeating digits. We then tried $\frac{1}{23}$. It had 22 repeating digits!*

- *We noticed that $\frac{1}{2}$ and $\frac{1}{8}$ both ended in zeroes when represented as fractions. We thought that maybe all fractions whose denominators were multiples of 2 would end in zeroes when turned into decimals. But that wasn't true. [Note: Any fraction whose denominator is of the form $2^a 5^b$, where a and b are integers greater than or equal to 0, terminates in zeroes. But this is not a fact students will likely discover after just one encounter with the sketch.]*

- *We wanted to see if there were any interesting patterns for fractions with a denominator of 11. We tried several fractions and they all had only two repeating digits. When we looked at which digits repeated, we found that 09 repeated for 1/11 and 18 repeated for 2/11. The digits in both cases added up to 9 (0 + 9 = 9 and 1 + 8 = 9). We guessed that $\frac{3}{11}$ might give us the repeating digits 27. We checked and we were right! The rest of the elevens followed the same pattern. Cool!*

- *We found that the fraction $\frac{1}{81}$ has a decimal expansion with all 10 digits in order: 0.1234567890123....*

There are many other patterns that students may notice depending on the fractions they have explored. Have students continue to share as time allows.

Credits:

This activity is based on a color calculator model developed by Nathalie Sinclair, a mathematics education at Simon Fraser University. You can read more about the color calculator in Sinclair's book, *Mathematics and Beauty: Aesthetic Approaches to Teaching Children* (Teachers College, 2006).

Related Activities:

- *Zooming Decimals—Exploring Tenths*

- *Zooming Decimals—Tenths, Hundredths, and Beyond*

- *Reach the Target—Decimals and Place Value*

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