Notes for the Teacher

Students use logical reasoning skills to arrange numbers and circles so that the sum of the numbers in each circle equals given target values. Forming the correct sums requires students to consider if and how the circles must overlap, with some numbers sitting in up to three circles simultaneously. Sketchpad generates random challenges for the students, or they can create their own problems to share.

Objectives:

- Given a collection of possible addends, students will select those numbers whose sum matches a given target value.
- Students will use logical reasoning to arrange sets of addends so that the numbers satisfy several conditions simultaneously.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (5) Use appropriate tools strategically; (7) Look for and make use of structure.

Common Core State Content Standards: 3.OA9

Grade Range: Grades 3–5

Introduce:

Open **Arranging Addends--Target Sum Puzzles.gsp** and distribute the worksheet. Use a projector to show sketch page "Two Circles."

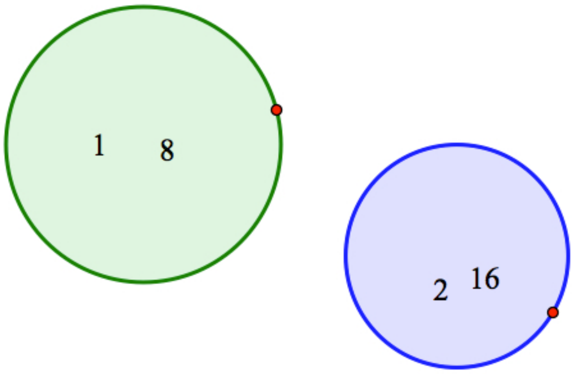
Read the directions aloud. Explain that the goal is to arrange the circles and the numbers so that the sum of the numbers in the green circle is 9 and the sum of the numbers in the blue circle is 18.

Demonstrate how to drag the numbers and the circles with the **Arrow** tool. Show how dragging a circle by its circumference moves the circle without changing its size. Show how dragging a circle by the point on its circumference causes the circle to either grow or shrink depending on whether the point is dragged towards the circle's center or away from it.

Do not drag the circles so that they overlap. Students will discover for themselves that in many Set Puzzles, the circles must overlap.

Give students time to think about the problem and then ask for responses. This first puzzle is simple. Have volunteers come to the computer and demonstrate their ideas.

Dragging the 1 and 8 into the green circle and the 2 and 16 into the blue circle solves the challenge. The solution is shown below. Students will observe that when the correct values are placed in the appropriate circles, both circles light up and the message "You got it!' appears.



Review the solution and how to record it on the worksheet. Some students may note that the solution to the puzzle did not require using all five numbers.

Circle	Numbers	Sum
Green	1, 8	9
Blue	2, 16	18
Red	—	—

Press *New Puzzle* with the **Arrow** tool to play again. Sketchpad will reset the circles and the numbers to their original positions and pick new random target sums for the two circles. (Note: It is possible that the required sum for a circle might be 1, 2, 4, 8, or 16. In these cases, discuss how only a single number needs to be dragged into the circle.)

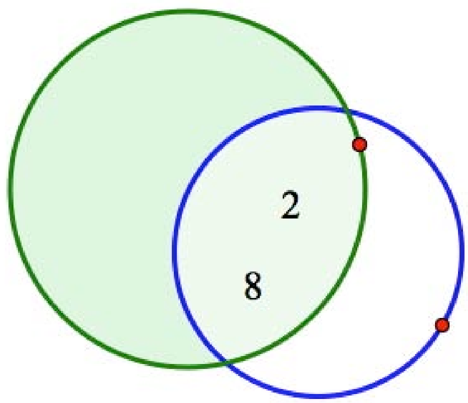
Suppose that the new puzzle states that the sum of the numbers in the green circle is 10 and the sum of the numbers in the blue circle is 11.

Ask, "Which numbers on our list have a sum of 10? Which numbers have a sum of 11?" Have volunteers come to the computer and demonstrate their ideas.

Students may make the following series of observations and deductions:

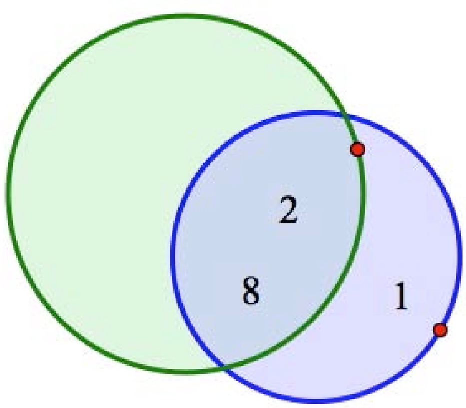
- *I'll drag the numbers 2 and 8 into the green circle since their sum is 10. No other combination of numbers on the list has a sum of 10. When I drag the numbers into the green circle, the inside of the circle lights up!*

- *The only way to make a sum of 11 from 1, 2, 4, 8, and 16 is to drag the 1, 2, and 8 into the blue circle. But 2 and 8 are already inside the green circle. If I drag them out of the green circle and place them in the blue circle, then the sum for the green circle will no longer be correct.*
- *I'm stumped! How can the 2 and 8 be in the green circle and the blue circle at the same time?*
- *Wait, I have an idea! Let me drag the circles so that they overlap. I'll arrange the numbers so that 2 and 8 sit in the overlapping portion of the circles. Now, the 2 and 8 are in both circles at the same time.*



- *We need to be careful where to put the 1. It needs to go into the blue circle so that we get a sum of 11, but it can't go into the portion of the circle that overlaps the green circle.*

The solution to the puzzle is shown below.

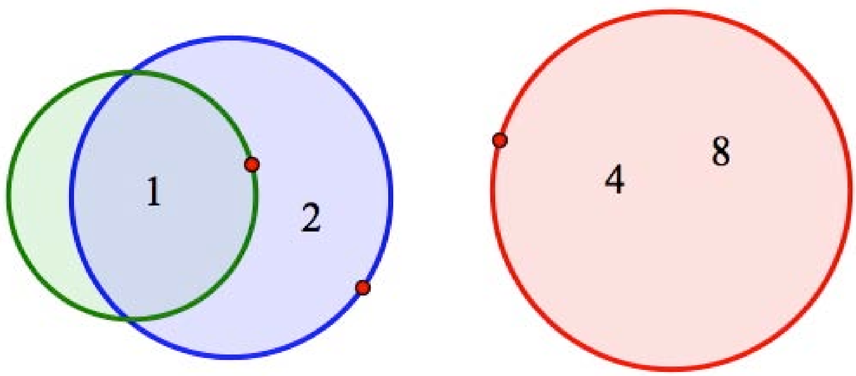


Review the solution and how to record it on the worksheet. You may wish to introduce the term *intersection* to describe the space where the green and blue circles overlap. You can say that the intersection is part of both circles.

Circle	Numbers	Sum
Green	2, 8	10
Blue	1, 2, 8	11
Red	—	—

When you're confident that students understand how the puzzles work, go to page "Three Circles." As before, read the directions aloud. There are now three circles instead of two, and each circle must be filled with numbers from the list 1, 2, 4, 8, 16, 32 to simultaneously satisfy the indicated sums.

The first challenge is fairly simple. Its solution is shown below.

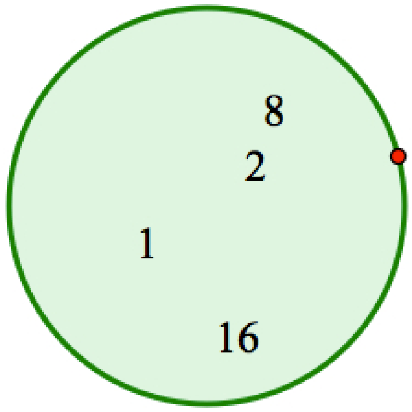


Circle	Numbers	Sum
Green	1	1
Blue	1, 2	3
Red	4, 8	12

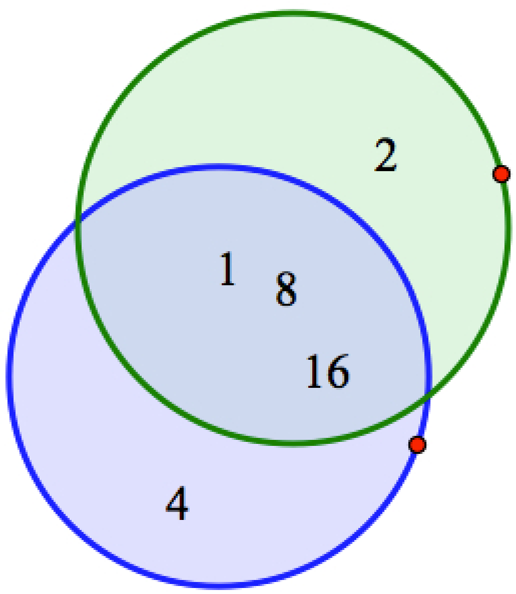
Here is another sample puzzle, harder than the one above, and a description of how a student might solve it:

> **Drag the numbers, circles, and points so that the sum of the numbers in the green circle is 27, the sum of the numbers in the blue circle is 29, and the sum of the numbers in the red circle is 42.**

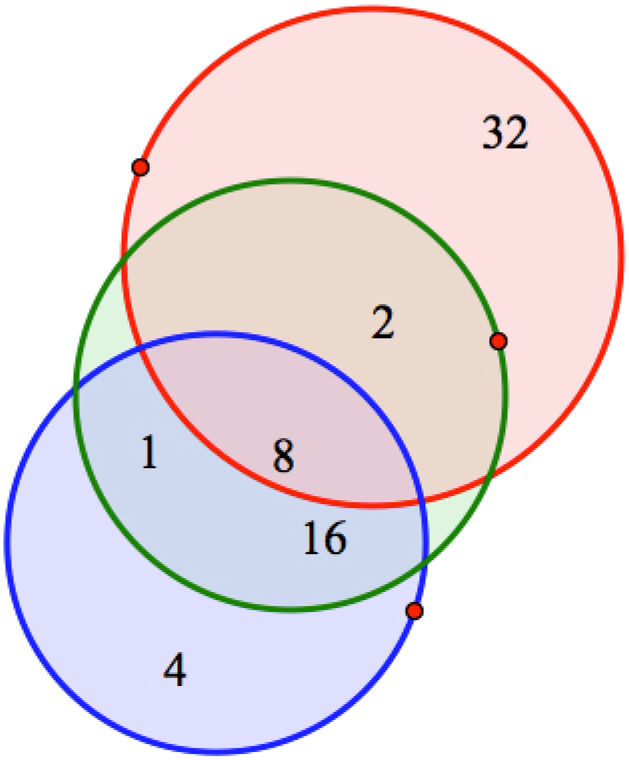
- *I know that 1 + 2 + 8 + 16 = 27, so I'll drag those numbers into the green circle.*



- *I know that 29 is 1 + 4 + 8 + 16. Hmmm...that means the 1, 8, and 16 are shared by the green and the blue circle. I'll have to move the blue circle so that it intersects the green circle. But I also need to move the numbers in my green circle so that just the 1, 8, and 16 lie in the intersection.*

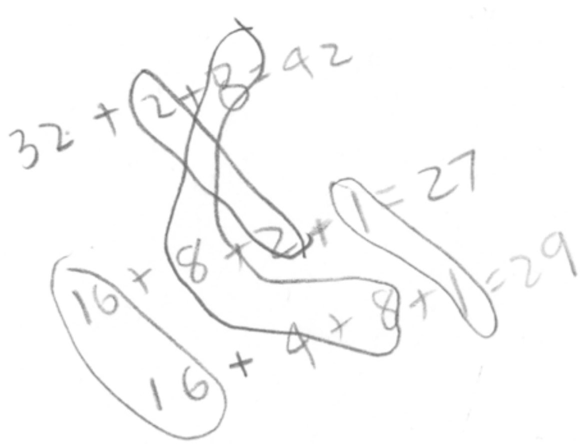


- *To get a sum of 42, I need to use 2, 8, and 32. That means the 8 is shared by all three circles. I need to move the number 1 a little to the left so that only the 8 lies in the intersection of all three circles.*



Circle	Numbers	Sum
Green	1, 2, 8, 16	27
Blue	1, 4, 8, 16	29
Red	2, 8, 32	42

Below is an example of actual student work for this puzzle:



The student found it helpful to write the sums on paper before dragging the numbers into the circles. Notice how the student circled the addends that were common to two or three of the sums. This helped her to determine which intersections, if any, each number should be placed in.

Explore:

Assign students to partners and send them in pairs to the computers. Have students open **Arranging Addends--Target Sum Puzzles.gsp** and go to page "Keeping Score" if you feel they're ready to play a scored game. If students need more time to practice, have them go to page "Two Circles" to practice with two circles or page "Three Circles" to practice with three circles.

Each game on the "Keeping Score" page consists of 10 random puzzles. If students solve a puzzle and press *Next Puzzle,* they'll receive 10 points for a correct answer and 0 points for an incorrect answer. If students press *Check Answer,* Sketchpad will tell them whether their solution is correct and allow them to fix their work if it is not. However, each incorrect check deducts 2 points from the value of the puzzle. So if a student checks her answer once and discovers it is incorrect, she will only receive 8 points for correcting her mistake. Nonetheless, when in doubt, checking one's work is a safer alternative than immediately pressing *Next Puzzle.*

You may want to allow students to use colored markers to make a quick sketch of each solution on blank paper as well, but be sure to tell them not to worry about being "perfect" when sketching the circles.

If there is extra time, encourage students to go to page "Make Your Own" and create their own puzzles to share. Review the steps on the worksheet to be sure students understand how to make a puzzle.

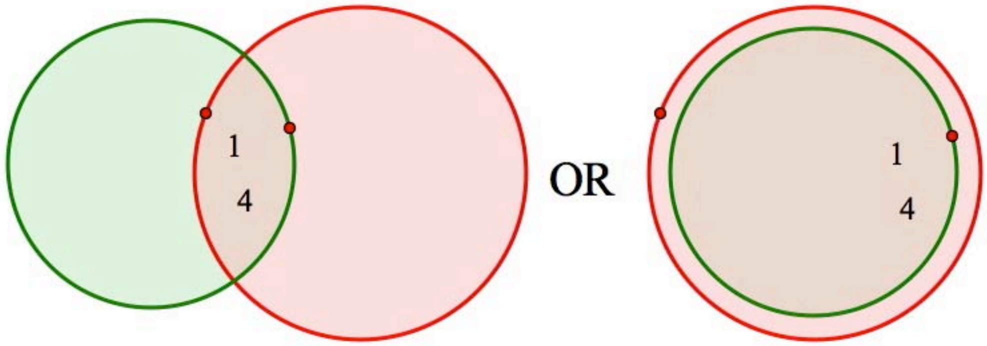
Challenge students to think about whether there is ever more than one solution to a puzzle. In fact, there is only one set of addends that sum to a number from 1 to 63 using only the numbers 1, 2, 4, 8, 16, and 32. The size of the circles and the placement of the numbers and circles can be arranged to appear superficially different, but these are all equivalent to each problem's one unique solution.

Discuss:

Call students together to discuss and summarize the different strategies used to solve the Set Puzzles. Here are some possible student observations:

- *If any sum was odd, we knew that we needed to use the number 1 to make it. All the other numbers available to us were even.*

- *We played a puzzle in which the sum of the numbers in the red and green circles both needed to be 5. We realized there were two ways to do this: We could either make the two circles partially intersect, or we could make either circle completely surround the other one.*



- *We discovered that if a sum was 32 or greater then the number 32 had to go inside that circle. If a sum was 16 or greater but less than 32, the number 16 had to go inside the circle. If a sum was 8 or greater but less than 16, the number 8 had to go inside the circle, and so on.*

- *It really helped us to write down the numbers on paper for each sum before we rearranged the numbers and circles. We could see which circles had the same numbers and needed to intersect.*

- *We found that each puzzle had only one group of numbers that worked for each circle. The circle sizes and placements might differ, but there was only one group of numbers that could go inside each circle.*

- *We noticed that each number in the list is equal to the previous number times 2. So 2 is equal to 1 × 2, 4 is equal to 2 × 2, 8 is equal to 4 × 2, and so on.*

- *It's amazing that whatever sum we are given, we can always find numbers from this list that add up to that sum. That's really cool! I never would have believed that every number from 1 to 63 can be formed using only the six addends, 1, 2, 4, 8, 16, and 32, each at most once.*

- *We think that if the game included 64 as one of the possible addends, we could form every sum from 1 to 127.*

At the end of the discussion, you may wish to mention the terms "Venn diagrams" and "sets" briefly, but the focus of the activity should squarely rest on honing students' reasoning skills.

A Mathematical Note for Your Information:

Why is it possible to form all the numbers between 1 and 63 using just six numbers—1, 2, 4, 8, 16, and 32—as addends?

Note that 1, 2, 4, 8, 16, and 32 are powers of 2. Let us examine how the numbers 15 and 21 are represented using these powers of two:

$$15 = \mathbf{1} \times 8 + \mathbf{1} \times 4 + \mathbf{1} \times 2 + \mathbf{1} \times 1$$

$$21 = \mathbf{1} \times 16 + \mathbf{0} \times 8 + \mathbf{1} \times 4 + \mathbf{0} \times 2 + \mathbf{1} \times 1$$

Written another way, we could say 15 = 1111 (base 2) and 21 = 10101 (base 2).

When students are using the numbers 1, 2, 4, 8, 16, and 32 to form the sums from 1 to 63, they are essentially finding the base 2 (binary) representations of these numbers. That is why it is possible to form every sum from 1 to 63 using this limited set of numbers and why each sum can only be formed in just one way.

Related Activities:

- *Cross Number Puzzles—Addition and Subtraction*
- *Mystery Sums, Part One—Deduce the Unknown Addends*
- *Mystery Sums, Part Two—Deduce the Unknown Addends*
- *Mystery Sums, Part Three—Dancing Addends*
- *Sneaky Sums—Unknowns in a Grid*

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