Notes for the Teacher

In this activity students use the independent and dependent variables from the previous activity (Special Effects—A Swirling Transformation) to define a Sketchpad **Swirl** command. Students use this command to apply the Swirl function to sets of input points (such as lines, circles, and pictures) to create the corresponding swirled output points. By animating the parameter that connects the original variables, students animate the function through a continuum of members of the swirling family, thus creating stunning visual effects. Finally, students are challenged to invent new special effects of their own.

This capstone activity rewards students' attention to the relative rate of change of variables, the correspondence between domain and range, the atomic and collective (mapping) views of function, the dual nature of functions both as actions performed on variables and as objects to be acted upon, and the membership of functions in families that share some characteristics and differ in others.

A new element in this activity is students' use of two related points to define a function that they can apply to pictures as well as geometric objects. Students' experiences are enriched by the use of picture transformations that map one region of the plane to another region. Because pictures are memorable, students can mentally compare an original picture with its image to visualize how the entire plane is transformed.

Expect the activity to take a single class period (about 45 minutes).

This is one of a series of Geometric Functions[1] activities in which students explore geometric transformations as functions. By using points as their independent and dependent variables, students can vary the independent variable and observe directly the behavior of the dependent variable. Students are encouraged to pay attention to the relative rate of change of the two variables and to other characteristics of the function (such as its fixed points). They trace the variables to record their locations over time (thus developing both *covariation* and *correspondence* views of a function), and they relate the shapes formed by the traces to their observations about relative rate of change and fixed points of the function. With this approach students directly manipulate variables to explore domain, range, composition, and inverse, making these concepts visible through dynamic images that reveal their fundamental aspects.

Objectives:

In this activity students will:

- Define a Swirl function using two related variable points.
- Apply the Swirl function to several geometric objects.
- Analyze, describe and explain the function by dragging the swirled objects.

[1] *Geometric Functions* (plural, capitalized) is used here to refer to this sequence of activities in which students explore geometric transformations as functions. A *geometric function* (lowercase) is used to refer to any transformation that takes a point to a point.

- Compare the user-defined **Swirl** command, which defines a function, with the built-in **Locus** command, which constructs the range that corresponds to a given domain.
- Apply the Swirl function to a picture and describe the results.
- Describe the connection between the atomic application of a function (to a single point) and the collective application (to an entire set of points).
- Create a special effect by animating the Swirl function through a continuum of members of the same family.
- Invent a new special effect function, apply it to a picture, and adjust the function until the desired effect is achieved.

Common Core Mathematical Practices: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning.

Common Core State Content Standards: 8.F1,2; 8.G1; F-IF1,2,9; G-CO2; G-SRT1

Grade Range: Grades 7–11

Prerequisites:

Before undertaking this activity, students must already have created a swirling function and animated the swirling effect by doing the following activities:

- *Compose a Locus—Composition, Domain, and Range*
- *Special Effects—A Swirling Transformation*

They should have saved their sketch from the *Swirling Transformation* activity to use for this current activity. Students should previously have either of the following:

- Three of the four function challenge activities (*Reflection Challenges, Rotation Challenges, Dilation Challenges, Translation Challenges), or*
- *Family Relationships—Rotation, Dilation, and Translation Families.*

These prior activities are also highly recommended:

- *ID the Suspects—Identify Functions*
- *Family Resemblances—Identify Function Families*

- *Dance the Dependent Variable—Geometric Function Dances*

Instructional Strategies:

This activity incorporates a number of instructional strategies designed to work together to develop students' conceptual understanding of functions.

Cognitive Demand: The short form of this activity contains high level directions focused on the mathematics, leaving the student responsible for figuring out the details of utilizing the software. The questions on both forms connect students' sensory-motor experiences to the mathematical objects they are manipulating and observing. (For instance, in Q2 students relate the differences between two Sketchpad commands to important aspects of various mathematical concepts.) The Challenge pages of **Animated Special Effects.gsp** provide open-ended tasks for students to explore.

Mathematical Habits of Mind, Reasoning and Sense Making: Students build the mathematical objects and then explore and analyze the behavior of these objects. The short form worksheet encourages students to be particularly self-reliant. Students must make mathematical sense of how the function maps an entire picture to a transformed version of the picture, and of how animating a parameter can modify the function through a continuum of members of a function family.

Inquiry: The worksheet contains probing questions that require students to manipulate, observe, and analyze. The Explore More questions expect students to apply the special effect to more images, and then to create new special effects of their own.

Cooperative Learning: Students work in pairs throughout the activity. Expect students to work purposefully together, to coach each other, and to discuss every part of the activity with their partners.

Assessment: You should assess student understanding by visiting and questioning student pairs, not only observing their work but also encouraging and guiding them. Use probing questions to encourage students to assess their own work and their own explanations. Use the summary discussion to probe students' depth of understanding and possible misconceptions. The last page of the worksheet is an exit ticket.

Differentiation: Different levels of students are supported by the worksheet, which is available in a long form (containing detailed Sketchpad instructions) and short form (containing only a mathematical overview of the objects to be constructed). Student pairs work together, coaching each other and consulting with other pairs for additional support. The optional Explore More questions are designed to engage students with different levels of background knowledge and to encourage self-directed work, and the Challenge pages of **Animated Special Effects.gsp** provide interesting open-ended tasks for students to do on their own.

Questioning and Discourse: Since most discourse will take place between and among pairs during inquiry, it's important to encourage students to describe and explain their construction methods and their observations to each other. Use the summary discussion to focus students' thinking on the big ideas, and the role that mathematical ideas play in the activity. These concepts include functions as mappings from domain to range, and functions as objects that can themselves be transformed through a continuum of members of a function family.

Instructional Strategies: By varying the independent variable, and later animating parameters to transform functions themselves through a set of related family members, students are already investigating similarities (what stays the same) and differences (what changes). This activity also makes strong use of multiple representations, conjecturing and testing hypotheses, and feedback that doesn't depend on the teacher.

Preparation:

Prepare by printing copies of the worksheet (**Animated Special Effects Worksheet.pdf**) for your class. Note that the worksheet is available in long and short forms; the forms contain the same questions, but the long form contains more detailed instructions. The short form is on a higher cognitive level, and gives students more responsibility for figuring out the details. It concentrates on the mathematical objects to be constructed and manipulated, omitting many of the simpler construction details. Most students who have completed the prerequisite activities should be ready to use the short form, though they may need to use the Sketchpad Tips (**Help | Using Sketchpad**) or the Reference Center (**Help | Reference Center**) to remind themselves of some techniques. Consider passing out the short form to all the teams and printing a few copies of the long form for students to consult as needed.

The worksheet pdf also includes a two-page answer sheet that lists all the questions from the worksheet with space for students to write their answers. An exit ticket is also included that you can pass out near the end of class to provide you with feedback while provoking students to think about what they've learned.

Review the two sketches that accompany the activity. Students do their work in a new sketch, but these sketches can be useful for other purposes. **Animated Special Effects.gsp** contains two pictures students that students will copy, paste into their own sketch, and use as the input for their Swirl function. The same sketch provides four challenges that ask students to define novel functions, apply them to pictures, and produce animated special effects. These challenges can serve as extra credit or as a follow-up activity for an advanced class. **Animated Special Effects Work.gsp** shows a typical construction at various stages identified by the numbered steps of the long form of the worksheet, and contains sample solutions for all of the Challenges.

Launch

Expect to spend about 5 minutes.

Explain to students that in today's activity they will start with the Special Effects sketch from the previous activity. They will use the Swirl function's variables to define the function in a way that makes it possible to apply the function much more easily and widely. Tell students that this new way of defining and applying functions can improve their understanding of the function concept, because it better relates the function in Sketchpad to the abstract mathematical concept of function.

Have a student volunteer open **Animated Special Effects Work.gsp,** go to page Define Custom Transform, and follow your direction. First the student drags independent variable *x* to verify that this page contains a 45° rotation function. Then the student selects both variables and chooses **Transform | Define Custom Transform.** (Remind students that *transformation* and *function* are mathematical synonyms, so this command is defining a rotation function for Sketchpad.) The proposed name for the function is "x → R[C,θ](x) Transform"; the student shortens the name to "R[C,θ]." Finally the student tests the new function by selecting the butterfly picture and choosing the **Transform | R[C,θ]** command to apply the function to the entire picture: every point of the picture is immediately transformed.

Ask students to explain the connection between the independent and dependent variable points and the two pictures of the butterfly. As part of the discussion, the student volunteer can drag *x* to a point on the original butterfly to verify that $R_{C,\theta}(x)$ goes to the corresponding point on the rotated butterfly. Encourage several students to explain, in their own words, how every single point of the original butterfly has been rotated, and that all of these rotated points work together to create the rotated butterfly.

Pass out the worksheets to the class. If you use the short form of the worksheet, tell students that they will be expected to figure out the construction details themselves by experimenting, by consulting with their partners and classmates, or by using a copy of the long form that will be made available to them. Students should use the Sketchpad Tips or the Reference Center for additional support. Remind students that each pair will have an operator (using the keyboard and mouse) and a coach (making suggestions and recording notes). Tell them to switch roles right after they answer Q4.

Remind students to monitor their own understanding of function concepts, and to keep track of what they understand well and what they're unsure about.

Explore

Expect to spend about 30 minutes.

Circulate as students work, making sure they write clear descriptions in complete sentences of the behavior they observe.

Question Q2 draws students' attention to the subtle but important differences between the **Swirl[C,θ]** command and the **Locus** command.

Question Q3 suggests a way of understanding the connection between the atomic view of functions, in which the function acts on a single location of the independent variable at a time, and the collective view, in which the function transforms an entire set of points (such as a polygon, circle, or picture) all at once.

Question Q5 uses the context of transforming a picture to revisit the connection between the atomic and the collective approaches to applying a function.

Be sure to call students' work to a halt early enough to conduct a summary discussion.

Summarize

Expect to spend about 10 minutes.

Students will undoubtedly have important observations to share as well as questions about their work. Many of the questions on the worksheet (particularly Q2, Q3, and Q6) deserve special attention.

Be sure to use this class summary discussion to revisit the big ideas of function, and to ask students how today's activity either gave new insights or modified existing beliefs about how functions behave. The Big Ideas page of **Animated Special Effects Work.gsp** contains bullet points summarizing some important ideas from this activity, and may be a useful way to end the discussion.

- *I can edit a function by changing the way the dependent variable is constructed (or calculated, for a numeric function).*
- *The relative rate of change determines the function's behavior (covariation).*
- *I can use the mapping from domain to range to understand and control a function's behavior (correspondence).*
- *I can apply the function to map an entire set of input points (a line, a circle, or a picture) to the corresponding output points.*
- *I can predict the behavior of other members of the function family if I understand the construction.*
- *I can animate a function to produce many family members and display a visual "special effect."*

Assess

Before class ends, ask students to fill out the exit ticket. Their answers can help you evaluate what students took away from the activity. Tell students the exit ticket can also be used for students to write their own questions or to describe their own difficulties regarding the activity.

Answers:

All answers should be in students' own words. Students are likely to make observations that contain both insights and misconceptions at the same time. Put more emphasis on the insights. Trying too hard to correct misconceptions can sometimes emphasize and perpetuate them. Instead, it's better if students can correct their own misconceptions by responding to probing questions or by listening to other students.

Q1 The transformed ray looks like the range created by the **Locus** command in the last activity. It's a spiral that is rotated by a greater and greater angle as it goes farther from the center.

Q2 Both **Swirl** and **Locus** can construct the range that corresponds to any given domain. But **Locus** requires a point that's already on a path, so it can't take just any example of a function. It requires three objects: an independent variable, a domain to which the variable is already restricted, and a dependent variable. But the **Swirl** command can be created from any example of related variables, and can be applied to just about any set of points, whether they form a path like a line or circle, or whether they are a region of the plane like a picture. (In this respect the **Swirl** is more general, more abstract—a better model of what a function really is.)

Q3 Students' explanations of the shape will vary, but should concentrate on how the distance from the center affects the angle of rotation. Ideally students should describe how their initial thoughts about what features in the range corresponded to certain parts of the domain were mistaken, and how they used their understanding of the function's behavior to develop a clearer explanation.

Q4 When the circular domain is centered at the origin, the range is also a circle. This is best explained by pointing out that rotating any point on the circle will leave it on the circle, so that the image of any circle centered at C is that same circle. (This does not mean, however, that any of these points are fixed points!)

Q5 The **Swirl** function has rotated each pixel of the original picture by an angle that depends on its distance from the center point. The result is a new picture in which the points farther from the center are rotated by larger angles, as seen in the modified shapes of the sectors.

Q6 Animating k causes the swirling effect to be more or less extreme, varying from no effect at all when $k = 0°$ to much more extreme swirling, with far greater curvature of the lines separating the sectors, when $k = 90$ or $k = –90°$.

Q7 Chosen pictures will vary, and students' descriptions will vary.

Q8 The special effects that students create will vary greatly.

Related Activities:

- *ID the Suspects—Identify Functions*
- *Family Resemblances—Identify Function Families*
- *Reflection Challenges—The Reflection Family*
- *Rotation Challenges—The Rotation Family*
- *Dilation Challenges—The Dilation Family*
- *Translation Challenges—The Translation Family*
- *Family Relationships—Rotation, Translation, and Translation Families*
- *Dance the Dependent Variable—Geometric Function Dances*
- *Transform Twice—Function Composition*
- *Compose a Locus—Composition, Domain, and Range*
- *Special Effects—A Swirling Transformation*

License (CC-BY-NC-SA 3.0)

Portions of this material are based upon work supported by the National Science Foundation under award number DRL-0918733. Any opinions, findings, and conclusions or recommendations expressed in this work are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.